SIMON FRASER UNIVERSITY

DEPARTMENT OF MATHEMATICS

Midterm: Solutions

MATH 240 Summer 2016

Instructor: Joep Evers

Wednesday June 22, 2016, 11:30am-12:20pm

Name: ______________________________ (please print)

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Instructions:

1. Do not open this exam until told to do so.
2. Write your name above in block letters. Write your SFU student number and email ID on the line provided for it.
3. Write your answer in the space provided below the question. If additional space is needed then use the back of the previous page.
4. **Explain your methods**, unless it is explicitly stated that no explanation is needed.
5. This exam has 5 questions on 6 pages (not including this cover page). Once the exam begins please check to make sure your exam is complete.
6. **No** calculators are allowed. **No** books, papers, or electronic devices shall be within the reach of a student during the examination. (**Cellphones off** please.)
7. This exam is administered under the SFU Code of Academic Honesty. **During the examination, communicating with, or deliberately exposing written papers to the view of other examinees is forbidden.**

Question	Points	Score
1	10	
2	4	
3	10	
4	5	
5	6	
Total	35	

[10] **1.** Indicate true or false. No further explanation needed. You will get 1 point per correct answer, 0 points for an incorrect or no answer.

T (a) Multiplication by the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ rotates vectors in $\mathbb{R}^2$ anticlockwise through an angle of 90°.

F (b) If A and B are invertible $n \times n$ matrices, then $(A+B)^{-1} = B^{-1} + A^{-1}$.

T (c) If A is an $m \times n$ matrix, $\mathbf{b}$ a vector in $\mathbb{R}^m$ and $\mathbf{p}$ a vector in $\mathbb{R}^n$ such that $A\mathbf{p} = \mathbf{b}$, then every solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ can be written as $\mathbf{x} = \mathbf{p} + \mathbf{w}$, where $\mathbf{w}$ is some solution of $A\mathbf{w} = \mathbf{0}$.

T (d) Any linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ that is one-to-one must also be onto.

T (e) The columns of any 3×4 matrix are linearly dependent.

T (f) A subspace of a vector space contains at least one element.

F (g) The null space of an $m \times n$ matrix A is a subset of $\mathbb{R}^m$.

T (h) Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for the vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_B$ from V to $\mathbb{R}^n$ is a one-to-one linear transformation.

T (i) If A is an invertible $n \times n$ matrix then $\dim \operatorname{Nul}(A) = 0$.

T (j) The rank of a matrix is the dimension of its row space.

[4] **2.** Complete the following definition:
Let V be a vector space and $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ a set of vectors in V. Then B is a basis for V if . . .

1. The set of vectors B is linearly independent; **and**

2. The set B spans V. That is, $\text{Span}\,\{\mathbf{b}_1, \ldots, \mathbf{b}_p\} = V$.

3. Throughout parts (a), (b), (c) and (d) of this question, let the matrix B and the vector $\mathbf{v}$ be given by

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 2 & -1 & 1 & -2 \\ 3 & 1 & 4 & 2 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 5 \\ 2 \\ 0 \\ 5 \end{bmatrix}$$

[2] (a) Find a such that the vector $\mathbf{x} = \begin{bmatrix} a \\ a \\ a \\ a \end{bmatrix}$ is a solution of $B\mathbf{x} = \mathbf{v}$.

Note that $\mathbf{x} = a \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Hence, $B\mathbf{x}$ is equal to:

$$B\mathbf{x} = a \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 2 & -1 & 1 & -2 \\ 3 & 1 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = a \begin{bmatrix} 10 \\ 4 \\ 0 \\ 10 \end{bmatrix}.$$

From this right-hand side, we conclude that $B\mathbf{x}$ equals $\mathbf{v}$ only for $a = 1/2$.

[5] (b) Find the solution of the homogeneous equation $B\mathbf{x} = \mathbf{0}$. If there are free variables, express the solution in parametric vector form.

Row reduce the augmented matrix $[\,B\,|\,\mathbf{0}\,]$:

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 2 & -1 & 1 & -2 & 0 \\ 3 & 1 & 4 & 2 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 \leftarrow R_3 - 2\cdot R_1 \\ R_4 \leftarrow R_4 - 3\cdot R_1 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & -5 & -5 & -10 & 0 \\ 0 & -5 & -5 & -10 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 \leftarrow R_3 + 5\cdot R_2 \\ R_4 \leftarrow R_4 + 5\cdot R_2 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 \leftarrow R_1 - 2\cdot R_2 \\ \\ \\ \\ \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

This reduced augmented matrix represents a consistent system of (effectively) two linear equations: $x_1 + x_3 = 0$ and $x_2 + x_3 + 2\,x_4 = 0$. There are two free variables. Introduce the parameters s and t in $\mathbb{R}$, such that $x_3 = s$ and $x_4 = t$. Then $x_1 = -s$ and $x_2 = -s - 2\,t$. The solution of the homogeneous equation is therefore

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s \\ -s - 2\,t \\ s \\ t \end{bmatrix}, \quad \text{for } s,\, t \in \mathbb{R}.$$

Expressed in parametric vector form, this is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \quad \text{for } s,\, t \in \mathbb{R}.$$

[1] (c) Is the matrix B invertible? Explain (shortly) why/why not.

No, B is not invertible.
An $n \times n$ matrix B is invertible if and only if $\mathrm{Nul}(B) = \{\mathbf{0}\}$. It follows from Question 3(b) that the homogeneous equation $B\mathbf{x} = \mathbf{0}$ has infinitely many solutions: $\dim \mathrm{Nul}(B) = 2$. Thus, B cannot be invertible.

[2] (d) Give the solution of the equation $B\mathbf{x} = \mathbf{v}$ without doing any further row operations.

Every solution $\mathbf{x}$ to the equation $B\mathbf{x} = \mathbf{v}$ can be written as the solution from 3(a) plus one vector in the solution set of $B\mathbf{w} = \mathbf{0}$ that we calculated in 3(b). See also the (true) statement in Question 1(c).

In parametric vector form, the general solution to $B\mathbf{x} = \mathbf{v}$ is therefore

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \quad \text{for } s, t \in \mathbb{R}.$$

[5] **4.** Show that the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 3 \end{bmatrix}$ is invertible **and** give its inverse A^{-1}.

Row reduce the augmented matrix $[\,A\,|\,I\,]$ as follows:

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 3 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_1 \leftarrow R_1 - 2\cdot R_2 \\ \\ R_3 \leftarrow R_3 - R_2 \end{array} \sim \left[\begin{array}{ccc|ccc} 0 & -1 & -2 & 1 & -2 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 & -1 & 1 \end{array}\right] \begin{array}{c} R_2 \leftarrow R_2 + R_1 \\ R_3 \leftarrow R_3 + R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 0 & -1 & -2 & 1 & -2 & 0 \\ 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & -3 & 1 \end{array}\right] R_1 \leftarrow R_1 + R_3 \quad \sim \left[\begin{array}{ccc|ccc} 0 & 0 & -2 & 2 & -5 & 1 \\ 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & -3 & 1 \end{array}\right] R_1 \leftarrow (-1/2)\cdot R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & -1 & 5/2 & -1/2 \\ 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & -3 & 1 \end{array}\right] R_2 \leftarrow R_2 + R_1 \quad \sim \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & -1 & 5/2 & -1/2 \\ 1 & 0 & 0 & 0 & 3/2 & -1/2 \\ 0 & 1 & 0 & 1 & -3 & 1 \end{array}\right] \text{reorder rows}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 3/2 & -1/2 \\ 0 & 1 & 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & -1 & 5/2 & -1/2 \end{array}\right].$$

1. This row reduction shows us that $A \sim I$, so A is invertible.
2. The inverse is now given by the block on the right-hand side of the reduced matrix. So

$$A^{-1} = \begin{bmatrix} 0 & 3/2 & -1/2 \\ 1 & -3 & 1 \\ -1 & 5/2 & -1/2 \end{bmatrix}.$$

[6] **5.** Let P be the space of *all* polynomials.
The sum of two polynomials f and g is defined such that $(f+g)(x) = f(x) + g(x)$ for all $x \in \mathbb{R}$, and scalar multiplication is defined by $(c\,f)(x) = c\,f(x)$ for all $x \in \mathbb{R}$.
The zero element is the polynomial z such that $z(x) = 0$ for all $x \in \mathbb{R}$.

Given: P is a vector space (*do not prove this*).
Show that P_4, the space of all polynomials *up to degree* 4, is a **vector space**.

We first show that P_4 is a subspace of P. To that aim, we have to check:

1. that the zero element from P is also in P_4. It is trivial that z such that $z(x) = 0$ for all $x \in \mathbb{R}$ is a polynomial of degree $\leqslant 4$.

2. that P_4 is closed under addition. The sum of two polynomials $a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + a_4\,x^4$ and $b_0 + b_1\,x + b_2\,x^2 + b_3\,x^3 + b_4\,x^4$ in P_4 is

$$\begin{aligned}(a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + a_4\,x^4) + (b_0 + b_1\,x + b_2\,x^2 + b_3\,x^3 + b_4\,x^4)\\ = (a_0+b_0) + (a_1+b_1)\,x + (a_2+b_2)\,x^2 + (a_3+b_3)\,x^3 + (a_4+b_4)\,x^4,\end{aligned}$$

for all $x \in \mathbb{R}$, where the coefficients are real and might or might not be zero. The right-hand side is again a polynomial of degree $\leqslant 4$, so P_4 is closed under addition.

3. that P_4 is closed under scalar multiplication. The multiplication of a polynomial $a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + a_4\,x^4$ in P_4 by the scalar $c \in \mathbb{R}$ is

$$\begin{aligned}c\,(a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + a_4\,x^4)\\ = (c\,a_0) + (c\,a_1)\,x + (c\,a_2)\,x^2 + (c\,a_3)\,x^3 + (c\,a_4)\,x^4,\end{aligned}$$

for all $x \in \mathbb{R}$, where the coefficients are real and might or might not be zero. The right-hand side is again a polynomial of degree $\leqslant 4$, so P_4 is closed under scalar multiplication.

Conclusion: P_4 is a subspace of **the vector space** P. Any subspace of a vector space is a vector space itself, so P_4 is a vector space.