Math 303 Midterm Test 2, March 2014

Closed book exam, no calculators.

Brief explanation is required whenever it is not clear how answers are obtained.

The test has 4 questions and is out of 40.

Last Name: ____________________ First Name: ____________________

Student Number: ____

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

1. Consider a simple random walk $(S_0, S_1, \ldots)$ on $\mathbb{Z}$ with $S_0 = 0$. The position S_n at time n is the result of making independent identically distributed steps $X_1, X_2, \ldots, X_n$ and $S_n = X_1 + X_2 + \cdots + X_n$. Suppose X_1 is 1 or 0 or -1, each with probability $\frac{1}{3}$.

   Recall from front page that brief explanations are required for answers on this and all other problems.

   (a) (3 points) Let $r(k) = \mathbb{E}_0 e^{ikX_1}$. Find $r(k)$ as a function of k.

   Solution: $r(k) = \frac{1}{3}e^0 + \frac{1}{3}e^{ik} + \frac{1}{3}e^{-ik} = \frac{1}{3} + \frac{1}{3}e^{ik} + \frac{1}{3}e^{-ik}$

   (b) (4 points) Find $\mathbb{E}_0 e^{ikS_n}$ in terms of $r(k)$ and n.

   Solution:

$$\mathbb{E}_0 e^{ikS_n} = \mathbb{E}_0 e^{ikX_1 + \cdots + ikX_n} = \mathbb{E}_0 e^{ikX_1} \ldots \mathbb{E}_0 e^{ikX_n} = r(k)^n$$

(c) (3 points) What is $\int_{-\pi}^{\pi} \left(\mathbb{E}_0 e^{ikS_n}\right) dk$ in terms of the event that $S_n = 0$.

Solution: Interchange integral and expectation. Since

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikS_n}\, dk = I_{S_n=0}$$

we get $(2\pi)\mathbb{E}_0 I_{S_n=0}$ which is $(2\pi)\mathbb{P}(S_n = 0)$.

2. Consider the branching process in which an individual has 2 offspring with probability p and otherwise has no children. Initially there is one individual. Let X_n be the number of offspring in generation n.

(a) (2 points) Define the generating function $G_n(s)$ of X_n.

Solution: $G_n(s) = \mathbb{E}s^{X_n}$ or $G_n(s) = \mathbb{P}(X_n = 0) + \mathbb{P}(X_n = 1)s + \mathbb{P}(X_n = 2)s^2 + \dots$

(b) (2 points) Find the generating function of X_1 as an explicit function of s.

Solution: $G_1(s) = (1 - p) + 0s + ps^2$

(c) (4 points) Compute, as a function of p, the probability η that the branching process goes extinct.

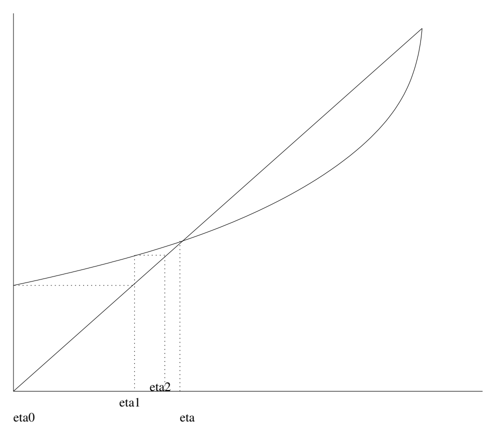
Solution: We have shown in class that the extinction probability is the smallest nonnegative root of $G(s) = s$ and $G = G_1$. Therefore we want the smallest nonnegative root of $(1 - p) + ps^2 = s$. One root is $s = 1$ and the other root is $s = \frac{1-p}{p}$. The extinction probability is therefore $\eta = \min(1, \frac{1-p}{p})$.

(d) (1 point) Let $p = \frac{1}{2}$ and let $\eta_n = \mathbb{P}(X_n = 0)$. Find η_1, η_2 and η_3 as rational numbers.

Solution: From definition of $G(s)$, $\eta_1 = G(0)$. Therefore $\eta_1 = \frac{1}{2}(1 + 0^2) = \frac{1}{2}$.
From class notes $\eta_2 = G(\eta_1) = \frac{1}{2}(1 + (\frac{1}{2})^2) = \frac{5}{8}$ and
$\eta_3 = G(\eta_2) = \frac{1}{2}(1 + (\frac{5}{8})^2)$

(e) (1 point) By a legible graph illustrate why η_n converges to the extinction probability.

Solution:



3. Let Q_{ij} be the transition matrix for simple random walk on $\mathbb{Z}$. Let $b_1 = 1, b_2 = 2, b_3 = 3$ and $b_i = 0$ for other integers i. Recall that the Metropolis-Hastings algorithm creates a Markov chain (X_n) on states $1, 2, 3$ by proposing a transition from i to j with probability Q_{ij} and accepting the proposal if $\frac{b_j Q_{ji}}{b_i Q_{ij}} \geq U$ where U is distributed uniformly in $[0, 1]$. Let P_{ij} be the transition matrix for (X_n).

(a) (3 points) What is the stationary probability row vector π for the Markov chain (X_n)?

Solution: As we have seen in homework $\pi = (cb_1, cb_2, cb_3)$ satisfies detailed balance where $c = \frac{1}{1+2+3}$ normalises, that is $\pi = (\frac{1}{6}, \frac{2}{6}, \frac{3}{6})$.

(b) (2 points) What is $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1,2,3} i^2 \mathbb{P}(X_n = i)$?

Solution: This problem was garbled. It should have read $\lim_{n\to\infty} \frac{1}{n} X_n^2$. According to the ergodic theorem this converges to $\sum_{i=1,2,3} i^2 \pi_i$ with π as above. In the exam I corrected it as shown above in red and this also converges as $n \to \infty$ to $\sum_{i=1,2,3} i^2 \pi_i = \frac{1}{6}1^2 + \frac{2}{6}2^2 + \frac{3}{6}3^2 = \frac{36}{6}$.

(c) (4 points) Find the 3×3 probability transition matrix P_{ij} explicitly.

Solution: As in homework, for $j \neq i$, $P_{ij} = Q_{ij} \min\left(\frac{b_j Q_{ji}}{b_i Q_{ij}}, 1\right) = \begin{bmatrix} & \frac{1}{2} & 0 \\ \frac{1}{4} & & \frac{1}{2} \\ 0 & \frac{1}{3} & \end{bmatrix}$ where $Q_{ij} = \frac{1}{2}$ for $j = i \pm 1$ and otherwise is zero. For diagonal entries we use the fact that the sum along the rows is one. $P_{ij} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$

(d) (1 point) Check if the stationary probability you claimed in part a is stationary for your answer in part c.

Solution: $(1, 2, 3) \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix} = (1, 2, 3)$ does hold, and multiplying everything by c does not change this.

4. Consider a bank with an ATM machine L on the left and a second ATM machine R on the right. Machine L processes customers in random $\exp(\lambda_L)$ times and machine R in random $\exp(\lambda_R)$ times. All times are independent. When you enter the bank both machines are busy and you are next in the line for whichever is first available.

   (a) (2 points) State, as an equation, the property of an exponential time T that makes it not matter how long customers ahead of you have been in line when you arrive.

   Solution: $\mathbb{P}(T > s + t | T > s) = \mathbb{P}(T > t)$.

   (b) (3 points) What is the joint probability density $f(s, t)$ for the exponential times of L and R for $s, t \geq 0$?

   Solution: $\lambda_L e^{-\lambda_L s} \lambda_R e^{-\lambda_R t}$

   (c) (4 points) Let T_L and T_R be the times that each machine takes on the customers ahead of you. Find the probability that you go to L by doing a double integral.

   Solution:

$$\begin{aligned}\mathbb{P}(T_L < T_R) &= \int_0^\infty \int_0^\infty I_{s<t} \lambda_L e^{-\lambda_L s} \lambda_R e^{-\lambda_R t}\, dt\, ds \\ &= \int_0^\infty \lambda_L e^{-\lambda_L s} \left(\int_s^\infty \lambda_R e^{-\lambda_R t}\, dt \right) ds \\ &= \int_0^\infty \lambda_L e^{-\lambda_L s} e^{-\lambda_R s}\, ds = \frac{\lambda_L}{\lambda_L + \lambda_R}\end{aligned}$$

(d) (1 point) Let $\lambda_R = 2\lambda_L$. What is the probability that you are still being served after both customers who were ahead of you have left?

Solution: The probability you go to L is $\frac{\lambda_L}{\lambda_L + 2\lambda_L} = \frac{1}{3}$. If you go to L then by the memoryless property the additional time that it takes to serve the customer at R is still $\exp(\lambda_R)$ so the answer is

$$\mathbb{P}(\text{go to } L)\mathbb{P}(T_L > T_R) + \mathbb{P}(\text{go to } R)\mathbb{P}(T_R > T_L)$$
$$= \frac{1}{3}\frac{2}{3} + \frac{2}{3}\frac{1}{3} = \frac{4}{9}.$$