Math 303 Midterm Test, February 2014

Closed book exam, no calculators.

Explanation is required whenever it is not clear how answers are obtained.

The test has 4 questions and is out of 40.

Last Name: ______________________ First Name: ______________________

Student Number: ____

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

1. Let $(X_0, X_1, \dots)$ be a time-homogeneous Markov chain.

   (a) (2 points) State the formula that defines the probability transition matrix P_{ij} in terms of $(X_0, X_1, \dots)$.

   Solution: $P_{ij} = \mathbb{P}\big(X_1 = j | X_0 = i\big)$.

   (b) (3 points) State the Markov property as equations relating conditional probabilities.

   Solution:

   $$\begin{aligned}\mathbb{P}\Big(X_{n+1} = j | X_0 = i_0, \dots, X_{n-1} = i_{n-1}, X_n = i\Big) \\ = \mathbb{P}\Big(X_{n+1} = j | X_n = i\Big).\end{aligned}$$

   These equations hold for all times n and all choices of states $i_0, \dots, i_{n-1}, i, j$.

   (c) (2 points) Express $\mathbb{P}(X_2 = j | X_0 = i)$ in terms of P.

   Solution: $\mathbb{P}\big(X_2 = j | X_0 = i\big) = \big(P^2\big)_{ij}$

(d) (3 points) Prove the formula you stated in part c using your answers to the previous parts of this equation.

Solution:

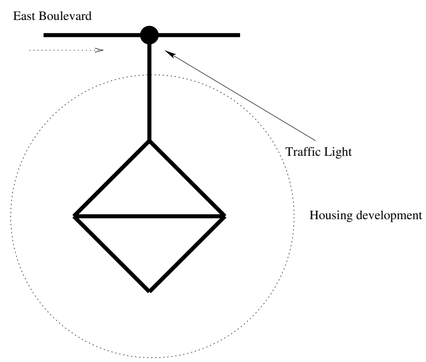
$$\mathbb{P}\big(X_2 = j | X_0 = i\big) = \sum_k \mathbb{P}\big(X_2 = j, X_1 = k | X_0 = i\big)$$

$$= \sum_k \mathbb{P}\big(X_2 = j | X_0 = i, X_1 = k\big)\mathbb{P}\big(X_1 = k | X_0 = i\big)$$

$$\overset{\text{part b}}{=} \sum_k \mathbb{P}\big(X_2 = j | X_1 = k\big)\mathbb{P}\big(X_1 = k | X_0 = i\big)$$

$$= \sum_k P_{kj} P_{ik} = \big(P^2\big)_{ij}$$

2. The map shows the roads in a housing development on a main road called East Boulevard. I drive along East Boulevard in the direction indicated by the dotted arrow and turn right at the traffic light into the housing project. At each subsequent intersection I make random choices from the available roads including U turns. Each road segment takes unit time to traverse.

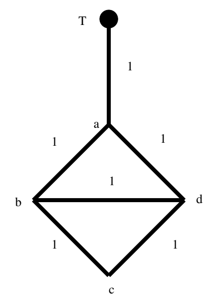


(a) (5 points) State the theorem that gives the stationary distribution for random walk on graphs with weighted edges. Do not forget to include all hypotheses, e.g. what kind of weights are allowed, etc.

> **Solution:** For a connected graph whose unoriented edges ij have positive weights $w_{ij} = w_{ji}$, for random walk on the graph with transition matrix $P_{ij} = \frac{w_{ij}}{\sum_k w_{ik}}$, there is a unique stationary distribution given by $\pi_i = \frac{\sum_k w_{ik}}{\sum_l \sum_m w_{lm}}$, provided the denominator is finite.

(b) (5 points) What is the expected time to return to the traffic light? Use the theorem in part a and begin by specifying how to choose the graph.

Solution: Apply the theorem to the connected weighted graph



For this graph $P_{Ta} = \frac{w_{Ta}}{w_{Ta}} = 1$ which corresponds to the fact that when I am first at the traffic light I traverse the edge into the housing development. I will not have the option to travel along East Boulevard until after I return to the traffic light so it is correct to remove it from the graph. The weights of one on each edge give the correct probability transition matrix that goes with making a random choice including a U turn at each intersection. Letting l, m run over all the vertices T, a, b, c, d we find that

$$\sum_{l,m} w_{lm} =$$

$$w_{Ta} + w_{aT} + w_{ab} + w_{ad} + w_{ba} + w_{bc} +$$
$$w_{bd} + w_{cb} + w_{cd} + w_{da} + w_{db} + w_{dc}$$
$$= 12.$$

By the theorem $\pi_{\text{traffic light}} = \frac{1}{12}$. This MC is irreducible. It has finitely many states so it is positively recurrent. By Theorem 4.1 the expected time to return is 12.

3. Let S_n be symmetric simple random walk on $\mathbb{Z}$ with $S_0 = 0$.

(a) (3 points) What is the definition of positive recurrence?

Solution: A recurrent state i is said to be positively recurrent if the expected time to return to i when starting in i is finite.

(b) (2 points) Give an example of an irreducible Markov chain that according to your definition in part a is not positively recurrent.

Solution: For (S_n) no state is positively recurrent. Since MC has to be irreducible finite transient chains like gamblers ruin are not right.

(c) (3 points) Give a formula in terms of j for $\mathbb{P}_0\big(S_{2j} = 4\big)$.

Solution: $\mathbb{P}_0\big(S_{2j} = 4\big) = \binom{2j}{j-2} 2^{-2j}$.

(d) (2 points) Give a formula in terms of binomial coefficients for $\mathbb{P}_0\big(S_{2j} = 4, E\big)$ where E is the event that $S_m \neq -1$ for $m = 0, 1, 2, \ldots, 2j$.

Solution: We subtract from the answer in part c the probability of walks that start at 0, hit state -1 before time $2j$ and are at state 4 at time $2j$.

Consider walks that start at 0, hit state -1 before time $2j$ and are at state 4 at time $2j$. For each such walk reflect the part of the walk before it first hits state -1. The reflection is such that each state $i \geq -1$ visited by the walk becomes state $-1-(i+1)$. This maps this set of walks one to one and onto the set of walks that start at -2 and at time $2j$ are at state 4.

The answer is answer to part c $- \mathbb{P}_{-2}\Big(S_{2j} = 4\Big)$, which is

$$\binom{2j}{j-2}2^{-2j} - \binom{2j}{j-3}2^{-2j}. \tag{1}$$

4. (a) (2 points) What equations define a stationary distribution π?

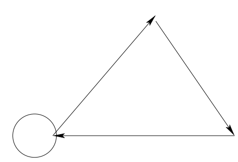
Solution: $\pi P = \pi$, $\sum \pi_i = 1$, all $\pi_i \geq 0$.

(b) (2 points) What are the equations of detailed balance?

Solution: $\pi_i P_{ij} = \pi_j P_{ji}$ for all states i, j, and $\sum \pi_i = 1$, all $\pi_i \geq 0$.

(c) (2 points) Must a stationary distribution satisfy the equations of detailed balance? Brief explanation or example required.

Solution:



No. In the picture the edges are one way: if $P_{ij} \neq 0$ then $P_{ji} = 0$. This contradicts having any π that solves $\pi_i P_{ij} = \pi_j P_{ji}$ and $\sum \pi_i = 1$. But by Thm 4.1 there is a unique π that solves $\pi P = \pi$.

(d) (2 points) For what kind of Markov chain does $\mathbb{P}_i(X_n = j)$ *not* have a limit as $n \to \infty$?

> **Solution:** Periodic. For example zerox machine with $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(e) (2 points) What is the probability that symmetric simple random walk starting at the origin reaches -1 before it reaches 9? Briefly explain your answer.

> **Solution:** This is $p = \frac{1}{2}$ gamblers ruin with states $0, 1, \ldots, 10$ but the states have been relabelled $-1, 0, \ldots, 9$. The answer is $\frac{9}{10}$ because state 0 is one tenth of the way from -1 to 9.