STATISTICS AND PROBABILITY

Table des matières

Chapitre 1

Probability

Main objective : understand the basic principles of probability

1.1 Introduction

Recall that inferential statistics involves taking a sample from a population, computing a statistic on the sample, and inferring from the statistic the value of the corresponding parameter of the population. The theory of probability is the basis for inferential statistics.

1.2 Methods of assigning probabilities

There are three methods of assigning probabilities

Classical method When probabilities are assigned based on laws and rules, the method is referred to as the classical method of assigning probabilities. It involves an experiment, which is a process that produces outcomes, and an event, which is an outcome of an experiment.

The probability of an individual event occuring is determined as the ratio of the number of items in a population containing the event (n_e) to the total number of items in the population (N) :

$$P(E) = \frac{n_e}{N}$$

n_e = the number of outcomes in which the event occurs out of N outcomes
N = total possible number of outcomes of an experiment.

Example : If a company has 200 workers and 70 are female, the probabilty of randomly selecting a female from this compagny is $\frac{70}{200} = 0.35$

Remark : Range of possible probabilities : $0 \leq P(E) \leq 1$

Relative frequecy of occurence The relative frequency of occurence method of assigning probabilities is based on cumulated historical data. With this method, the probability of an event occuring is equal to the number of times the event has occured in the past divided by the total number of opportunities for the event to have occured.

$$\frac{\text{Number of times of an Event occured}}{\text{Total Number of Opportunities for the event to occur}}$$

Example : A company wants to determine the probability that its inspectors will reject the next batch of raw materials from a supplier. Data gathered from company record books show that the supplier sent the company 90 batches in the past, and inspectors rejected 10 of them. By the method of relative frequency of occurence, the probability that the inspectors will reject the next batch is $\frac{10}{90} = 0,11$

If the next batch is rejected, the relative frequency of occurence probability for the subsequent shipment would change to $\frac{11}{91} = 0.12$

Subjective method The subjective method of assigning probability is based on the feelings or insights of the person determining the probability. Subjective probability comes from the person's intuition or reasoning. Although not a scientific approach to probability, the subjective method is often based on the accumulation of knowledge, understanding, and experience stored and processed in the human mind. At times it is merely a guess. At other times, subjective probability can potentially yield accurate probabilities.

Summary Subjective probabilities depend on person's knowledge and experience. So they can vary from person to person. Classical and relative probabilities depend on objective conditions. So they remain the same no matter who calculate them.

1.3 Structure of probabilities

Experiment : An experiement is a process that produces outcomes.

- Rolling a die
- Testing new pharmaceutical drugs on samples of cancer patients and measuring the patients' improvement.
- Sampling five bottles coming off a production line
- Auditing every 10th account to detect any errors
- Recording the Toronto Stock Exchange average on the first Monday of every month for 10 years
- Interviewing 20 randomly selected consumers and asking them which brand of appliance they prefer

Event : An event is an outcome of an experiment. The experiment then defines the possibilities of the event.

- If the experiment is to sample five bottles coming off a production line, an event could be to get one defective and four good bootles, two defective and 3 good bottles and so on...

- In an experiment to roll a die, one event could be to roll an even number

Elementary events : Events that cannot be decomposed or broken down into other events are called elementary events.

If the experiment is to roll a die, the elementary events are to roll a 1, or a 2, a 3, and so on. Rolling an even number is an event but is not an elementary event because the even number can be broken down further into events 2, 4, 6.

In the experiment of rolling a pair of die, there are 36 possible elementary events.

Other events : two even numbers, a sum of ten, a sum greater than 5, ...

Sample space A sample space is a complete roster or listing of all elementary events for an experiment.

The sample space for the rool a single die is $\{1, 2, 3, 4, 5, 6\}$
The sample space for the rool of a pair of dices is ...
Sample space can aid in finding probabilities. Suppose that an experiment is to roll a pair of dice. What is the probability that the dice will sum to 7 ?

An examination of the sample space (shown above) reveals that there are six outcomes in which the dice sum to 7. in the 36 total possible elementary events in the sample space. They are $\{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$

We can conclude that the mentioned probability is $6/36 = 0.1667$

Unions and Intersection Given two sets X, Y :

- The union of X, Y is formed by combining elements from both X and Y. The union expression $X \cup Y$ can be translated to X or Y.
- The intersection of X, Y denoted by $X \cap Y$ contains elements common to both sets. The intersection symbol $\cap$ is often read as *and*.

Example and representation : page 109

Mutually exclusive events Two or more events are mutually exclusive events if the occurence of one event precludes the occurence of the other event(s). This characteristic means that mutually exclusive events cannot occur simultaneously and therefore can have no intersection.

If X and Y are two mutually exclusive events then the probability of $X \cap Y$ is : $P(X \cap Y) = 0$.

In the experiment of the toss of a single coin, heads and tails are mutually exclusive events.

In a sample of manufactured products, the event of selecting a defective part is mutually exclusive with the event of selecting a nondefective part.

Independent events Two or more events are independent events if the occurence or nonoccurence of one of the events does not affect the occurence or nonoccurence of the other event(s).

Experiments such as rolling dice, yield independant events : each die is independent of the other. Whether a 6 is rolled on the first die has no influence on whether a 6 is rolled on the second die.

As well coin tosses are always independant of each other. The event of getting a head on the first toss of a coin is independent of getting a head in the second toss.

If X and Y are independent, then the probability that of X occuring given that Y has occured is just the probability of X occuring.

Independence of X and Y means that : $p(X|Y) = p(X)$ and $p(Y|X) = p(Y)$

Collectively exhaustive events A list of collectively exhaustive events contains all possible elementary events for an experiment. The sample space can be described as a list of events that are mutually exclusive and collectively exhaustive.

Complementary events The complement of event A is denoted A' (not A). It is the set of elementary events of an experiment not in A. For example in rolling a die, if event A is getting an even number, the complement of A is getting an odd number that is, $A = \{2, 4, 6\}$ and $A' = \{1, 3, 5\}$

Representation : page 111 figure 4.5.

Using the complement of an event can sometimes be helpful in solving probabilities because of the following rule.

$p(A') = 1 - p(A)$. **Example : Points of interest, page 111**.

Counting the possibilities - Determining the size of the sample space of an experiment

We present here four counting methods of number of outcomes that can occur for some particular experiments. These rules can delineate the size of the sample space.

- **The mn counting rule** : For an operation that can be done m ways and a second operation that can be done n ways, the two operations can then occur, in order, in mn ways. This rule can be extended to cases with three or more operations. **Examples :**
  - At the beginning of the mn Counting rule,
  - Fin de page 111 and Points of interest page 112.
- **Sampling from a population with replacement** : Sampling n items from a population of size N with replacement provides N^n possibilities where n is the sample size and N the population size.

Example : If a die is rolled three times in succession, how many different outcomes can occur ? That is, what is the size of the sample space for this experiment ?

- The size of the population is $N = 6$, the 6 sides of the die and the sample size is $n = 3$. The sample space is N^n.

- Suppose in a lottery six numbers are drawn from the digits 0 through 9, with replacement (digits can be reused). How many different groupings of six numbers can be drawn ?

N is the population of 10 numbers and n is the sample size, 6 numbers.

- **Combinations : Sampling from a population without Replacement** : The third method uses combinations or sampling without replacement. Sampling n items from a population of size N without replacement provides : ${}_NC_n = \binom{N}{n} = \frac{N!}{n!(N-n)!}$ where $n!$ is factorial n. $0! = 1$ and $n! = n(n-1)...3.2.1$

Example : page 112.

- **Counting the number of sequence** : The number of sequence of r elements in a set of n items is denoted and given by : ${}_nP_r = \frac{n!}{(n-r)!}$: n is the number of elements in the set and r : the number of elements to be selected from the set. When we use all members of a set in every sequence we obtain what is called a permutation. We therefore have ${}_nP_r = n!$.

1.4 Marginal, union, joint probabilities

- Marginal probability is denoted $P(E)$ where E is some event. A marginal probability is usually computed by dividing some subtotal by the whole.

An example is the probability that a person owns a Ford car. This probability is computed by dividing the number of Ford owners by the total number of car owners.

- Union probability is denoted $P(E_1 \cup E_2)$ where E_1 and E_2 are two events. It is the probability that E_1 will occur, or that E_2 will occur, or that both will occur. An example of union probability is the probability that a person owns a Ford or a Chevrolet. To qualify for the union the person has to have at least one of these cars.

- The joint probability of two events E_1 and E_2 occuring is denoted $P(E_1 \cap E_2)$. Sometimes it is read as the probability of E_1 and E_2. To qualify for the intersection, both events must occur.
- The conditional probability. Conditional probability is denoted $P(E_1|E_2)$. This expression is read as the probability that E_1 will occur given that E_2 is known to have occured. Conditional probabilities involve knowledge of some prior infomation.

Representation : page 115, figure 4.6.

1.5 Addition conditional probabilities and multiplication laws

- **The general law of addition** : The general law of addition is used to find the probability of the union of two events.

$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$ where X and Y are events, $X \cap Y$ the intersection of X and Y.

Example : Probleme page 131. (add the probability of the intersection, 56%)

- **The special law of addition** : If X and Y are mutually exclusive events, then $P(X \cup Y) = P(X) + P(Y)$
- **The complement of the union**. Recall that the probability union of two events X and Y represents the probability that the outcome is either X or Y or it is both X and Y.

Not (X or Y) is equivalent to Neither X nor Y

P(Neither X nor Y) $= P(notX \cap notY) = P(A' \cap B') = 1 - P(X \cup Y)$

Representation and Example : page 121 tout au debut.

- **The conditional probability** of X occuring given that Y is known or has occured is denoted and expressed as $P(X|Y) = \frac{P(X \cap Y)}{P(Y)}$

Example : page 131,

- **Probability matrix** : A probability matrix displays the marginal probabilities and the intersection probabilities of a given problem. Generally, a probability matrix is constructed as a two dimentional table with one variable on each side of the table.

For example in the office design problem, noice reduction would be on one side of the table and increased storage space on the other .

Construction of the Office Design problem :

	Increased storage space		
Noice reduction	0.56	0.14	0.7
	0.11	0.19	0.3
	0.67	0.33	

Example : page 118 Problem 4.1

- **Multiplication laws :** The general law of multiplication is used to find the joint probability.

$P(X \cap Y) = P(X).P(Y|X) = P(Y).P(X|Y)$.

$X \cap Y$ means that both X and Y must happen. The general law of multiplication gives the probability that both event X and Y will occur at the same time.

Example : Page 125 and

Example : Demonstration problem page 4.5 page 126

Remark : If X and Y are independent, then $P(X \cap Y) = P(X).P(Y)$
Example : Problem 4.7 page 128.
Example : Problem 4.2 page 119.

1.6 Revision of probabilities : Bayes' rule

Objective : Calculate conditional probability using the Bayes' rule.

Bayes' rule is a formula that extends the use of the law of conditional probabilities to allow revision of original probabilities with new information. It is therefore used to "revise" probabilities in light of new information.

Prior probabilities are based from the original information

Problem : Page 138 : A particular...

Bayes' rule :
$$\begin{aligned} P(X_i|Y) &= \frac{P(X_i) \times P(Y|X_i)}{P(Y)} \\ &= \frac{P(X_i) \times P(Y|X_i)}{P(X_1) \times P(Y|X_1) + P(X_2) \times P(Y|X_2) + ... + P(X_n) \times P(Y|X_n)} \end{aligned}$$

$P(X_i)$ is the prior probability and $P(X_i|Y)$ the posterior probability.

The denominator is referred to as the "total probability". It is a collectively exhaustive listing of mutually exclusive outcomes of Y. It represents a weighted average of the conditional probabilities, with the weights being the prior probabilities of the corresponding event.

Solving the **problem above** and **Another example** : problem 4.12 page 140