

# I. RELATIVITY

« Relativity of time depends on which side of the washroom door you are. »  
Anonymous

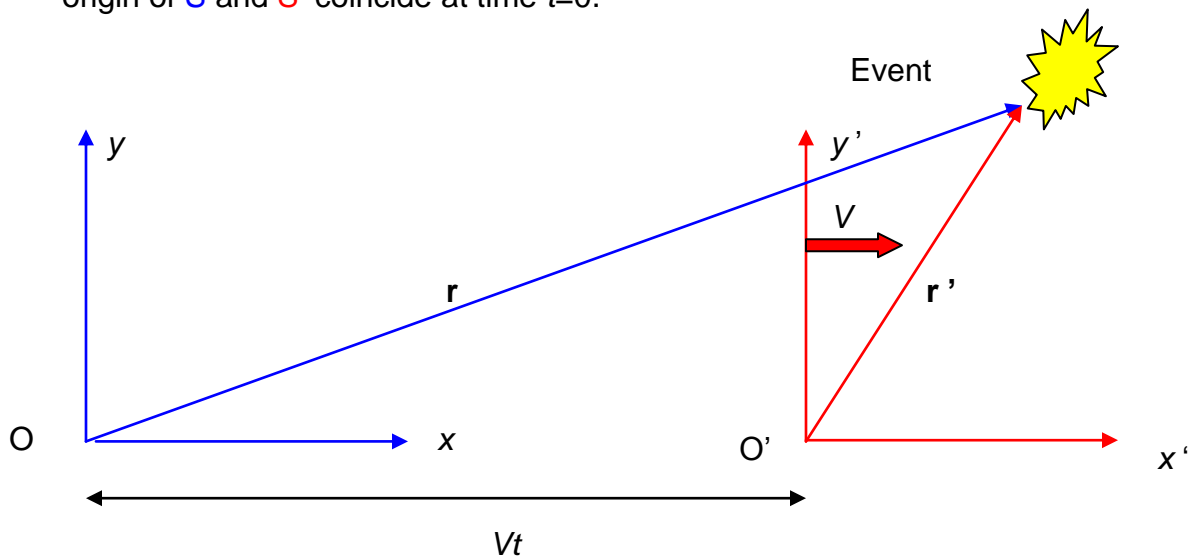
## 1.1 RELATIVISTIC KINEMATICS

### 1.1 Galilean relativity

Principle of classical relativity:

The laws of mechanics are covariant (i.e. have the same form in all inertial frame of reference).

Let a Cartesian system  $S'$  in uniform linear motion with velocity  $V$  with respect to another Cartesian frame  $S$ . Set the direction of  $V$  along the  $X$  axis and let the origin of  $S$  and  $S'$  coincide at time  $t=0$ .



Galilean transformation:

$$\begin{aligned}x' &= x - Vt \\y' &= y \\z' &= z \\t' &= t\end{aligned}$$

or again:

$$\mathbf{r}' = \mathbf{r} - \mathbf{V}t$$

Differentiating with respect to time, we find the transformation of velocities:

$$\mathbf{V}' = \mathbf{v} - \mathbf{V}$$

Examples of covariance:

Consider two bodies in mutual interaction with a force that depends on the distance between them.

a) Second law of Newton:

$$m_1 \frac{d^2 \mathbf{r}_1}{dt^2} = \mathbf{F}(|\mathbf{r}_2 - \mathbf{r}_1|).$$

Let's go to S':

$$m_1 \frac{d^2 \mathbf{r}'_1}{dt^2} = m_1 \frac{d^2}{dt^2} (\mathbf{r}_1 - \mathbf{V}t) = m_1 \frac{d^2 \mathbf{r}_1}{dt^2}.$$

And:  $|\mathbf{r}'_2 - \mathbf{r}'_1| = |\mathbf{r}_2 - \mathbf{V}t - \mathbf{r}_1 + \mathbf{V}t| = |\mathbf{r}_2 - \mathbf{r}_1|.$

One concludes:

$$m_1 \frac{d^2 \mathbf{r}'_1}{dt^2} = \mathbf{F}(|\mathbf{r}'_2 - \mathbf{r}'_1|).$$

b) Momentum conservation:

In a collision ( $i$ =initial,  $f$ =final):  $m_1 \mathbf{v}_1^i + m_2 \mathbf{v}_2^i = m_1 \mathbf{v}_1^f + m_2 \mathbf{v}_2^f.$

Let's go to S':

$$m_1 \mathbf{v}'_1^i + m_2 \mathbf{v}'_2^i = m_1 \mathbf{v}_1^i - m_1 \mathbf{V} + m_2 \mathbf{v}_2^i - m_2 \mathbf{V}.$$

$$m_1 \mathbf{v}'_1^f + m_2 \mathbf{v}'_2^f = m_1 \mathbf{v}_1^f - m_1 \mathbf{V} + m_2 \mathbf{v}_2^f - m_2 \mathbf{V}.$$

One concludes that the momentum is also conserved in S':

$$m_1 \mathbf{v}'_1^i + m_2 \mathbf{v}'_2^i = m_1 \mathbf{v}'_1^f + m_2 \mathbf{v}'_2^f.$$

c) Energy conservation in an elastic collision:

$$\frac{1}{2}m_1v_1^{i2} + \frac{1}{2}m_2v_2^{i2} = \frac{1}{2}m_1v_1^{f2} + \frac{1}{2}m_2v_2^{f2}.$$

Let's go to S':

$$\begin{aligned} \frac{1}{2}m_1v_1^{i'2} + \frac{1}{2}m_2v_2^{i'2} &= \frac{1}{2}m_1v_1^{i2} - m_1\mathbf{v}_1^i \cdot \mathbf{V} + \frac{1}{2}m_1V^2 + \frac{1}{2}m_2v_2^{i2} - m_2\mathbf{v}_2^i \cdot \mathbf{V} + \frac{1}{2}m_2V^2 \\ &= \frac{1}{2}m_1v_1^{f2} + \frac{1}{2}m_2v_2^{f2} - (m_1\mathbf{v}_1^f + m_2\mathbf{v}_2^f) \cdot \mathbf{V} + (m_1 + m_2)V^2/2 \\ &= \frac{1}{2}m_1v_1^{f'2} + \frac{1}{2}m_2v_2^{f'2} \end{aligned}$$

where one has used the conservation of energy and of momentum in S. One concludes that the energy is also conserved in S'.

Problem with classical relativity:

The laws of electromagnetism are NOT covariant in a Galilean transformation!

Ex. Consider a light flash along the X axis in S:



In S: The medium is at rest. The speed of light is c.

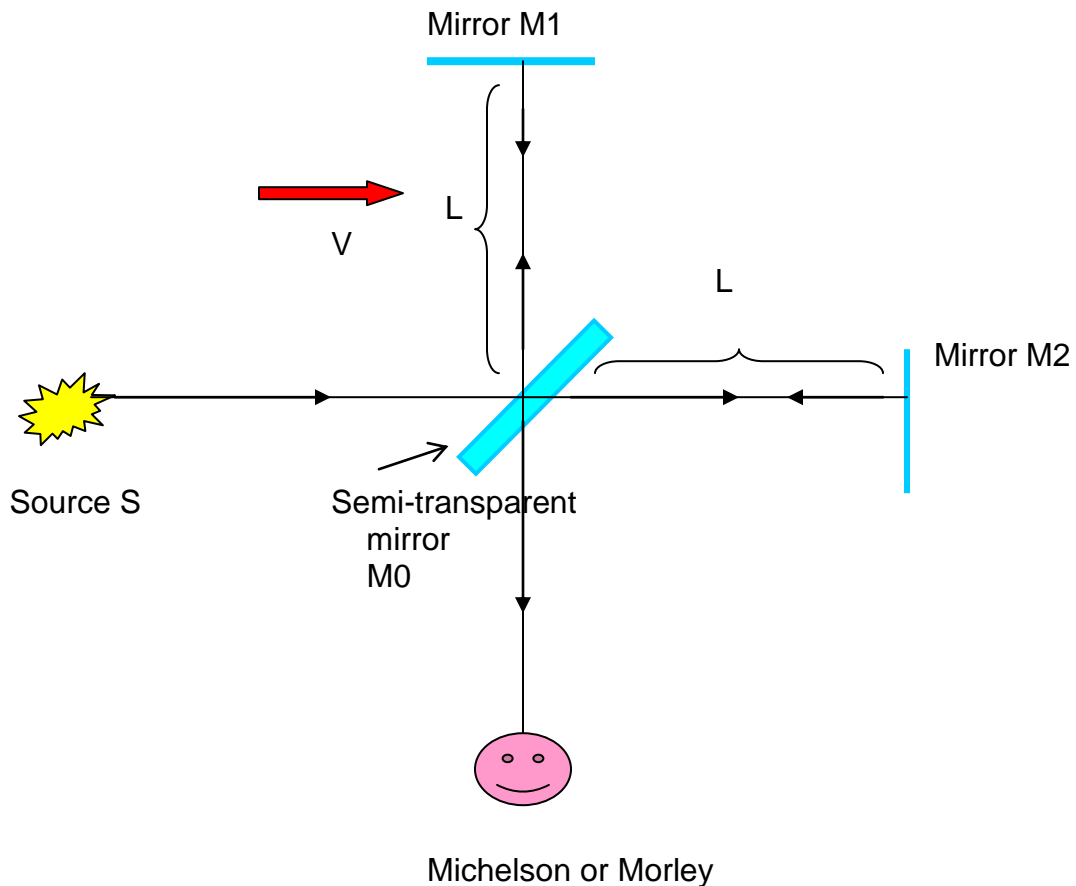
In S': Following Galileo, a measurement of the speed of light would give c-V.

The speed of light would depend on the motion of S' with respect to an "absolute" frame of reference.

Classically, there would a system S that is not equivalent to the others: this is the one for which the medium in which light propagates is at rest (the infamous « ether »).

## 1.2 The Michelson-Morley experience (1887)

Object: to detect the motion of the Earth through ether.

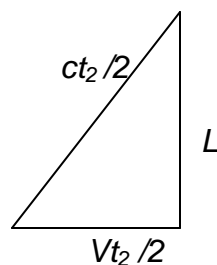


Time of flight along the path M0-M2-M0:

$$t_1 = \frac{L}{c-V} + \frac{L}{c+V} = \frac{2Lc}{c^2 - V^2}$$

Time of flight along the path M0-M1-M0:

$$\frac{1}{4}c^2 t_2^2 = L^2 + \frac{1}{4}V^2 t_2^2 \Rightarrow t_2 = \frac{2L}{\sqrt{c^2 - V^2}}$$



The phase difference that Michelson or Morley would observe is ( $\lambda$ = wavelength):

$$\Delta\phi = \frac{2\pi}{\lambda} c(t_1 - t_2) = \frac{2\pi L}{\lambda} \left( \frac{2}{1 - V^2/c^2} - \frac{2}{\sqrt{1 - V^2/c^2}} \right)$$

With  $V \cong 30$  km/s,  $V/c \ll 1$ , this is:

$$\frac{\Delta\phi}{2\pi} \cong \frac{2L}{\lambda} \left( \frac{V^2}{c^2} - \frac{V^2}{2c^2} \right) = \frac{LV^2}{\lambda c^2}.$$

In rotating the whole apparatus by  $90^\circ$ , one should observe a relative phase shift

of: 
$$\frac{\Delta\phi_{tot}}{2\pi} = \frac{2LV^2}{\lambda c^2}.$$

With  $L = 11$  m,  $\lambda = 500$  nm,  $\frac{\Delta\phi_{tot}}{2\pi} = 0.4$ , which can be easily detected.

Result:  $\Delta\phi = 0$ , nada, zilch, nothing, rien.

Consequences:

- The speed of light is always  $c$  in all inertial frames of reference.
- The Galilean transformation must be modified.
- A new conception of space and time is necessary.
- Specifically, the « obvious » hypothesis  $t' = t$  must be revised.

### 1.3 Postulates of special relativity (Einstein)



Albert Einstein (1879-1955)  
Swiss Physicist  
Nobel Prize (1921) for the  
theory of the photoelectric  
effect

« Einstein on the beach » (opera by Philip Glass)

A. The laws of nature (mechanics and electromagnetism) are covariant in all inertial frames of reference.

B. In all inertial frames, the speed of light in vacuum is the same in all direction ( $= c$ ).

## 1.4 Lorentz transformation

Let  $S'$  be a frame of reference in uniform linear motion with velocity  $V$  measured in a frame  $S$ .  $V$  is along the  $x$  axis. We desire to relate  $x', t'$  to  $x, t$ . The origin of both  $S$  and  $S'$  coincide at  $t=t'=0$ . A general linear transformation can be written as:

$$x' = \gamma x + Ft$$

$$y' = y$$

$$z' = z$$

$$t' = At + Bx.$$

We need to find the constants  $A, B, F, \gamma$ . We want to make sure that the Galilean transformation is recovered for  $V/c \rightarrow 0$ . The origin coincide at  $t=t' = 0$ . Hence,  $x'=0$  (the origin of  $S'$ ) corresponds to  $x=Vt$ :

$$\begin{aligned} 0 &= (\gamma V + F)t \Rightarrow F = -\gamma V \\ &\Rightarrow x' = \gamma(x - Vt) \end{aligned}$$

We therefore need  $\gamma \rightarrow 1$  when  $V/c \ll 1$ . Similarly,  $A \rightarrow 1$  and  $B \rightarrow 0$  when  $V \rightarrow 0$ .

- Consider a light flash emitted at the origin of  $S$ . The position of the wavefront in  $S$  is:

$$c^2 t^2 - x^2 - y^2 - z^2 = 0.$$

Use the second postulate. The wavefront in  $S'$  is:

$$c^2 t'^2 - x'^2 - y'^2 - z'^2 = 0.$$

- Use the linear form written above:

$$c^2 t^2 - x^2 = c^2 A^2 t^2 + 2c^2 ABtx + c^2 B^2 x^2 - \gamma^2 x^2 + 2\gamma^2 Vtx - \gamma^2 V^2 t^2$$

Compare the coefficients of  $x^2$ ,  $xt$  and  $t^2$ :

$$c^2 B^2 - \gamma^2 = -1; \quad \gamma^2 V + c^2 AB = 0; \quad c^2 A^2 - \gamma^2 V^2 = c^2.$$

This gives:

$$B^2 = (\gamma^2 - 1)/c^2; \quad A^2 B^2 = \gamma^4 V^2 / c^4; \quad A^2 = 1 + \gamma^2 V^2 / c^2.$$

Or again:  $\gamma^4 V^2 / c^4 = (1 + \gamma^2 V^2 / c^2)(\gamma^2 - 1)/c^2$ . Hence:  $\gamma^2(1 - V^2/c^2) = 1$ .

The final result is then:  $\gamma = 1/\sqrt{1 - V^2/c^2} = A; \quad B = -\gamma V/c^2$ .

The  $\gamma$  factor is a well-known relativistic correction term. To avoid confusion as to what velocity is used in the definition of this factor, we will add a subscript  $V$  to  $\gamma$ . The relativistic Lorentz transformation is then:

$$x' = \gamma_V(x - Vt); \quad y' = y; \quad z' = z; \quad t' = \gamma_V(t - Vx/c^2).$$
$$\gamma_V = 1/(1 - V^2/c^2)^{1/2}.$$

The inverse transformation (from  $S'$  to  $S$ ) can be found by interchanging the role of the coordinates and in setting  $V \rightarrow -V$ :

$$x = \gamma_V(x' + Vt'); \quad y = y'; \quad z = z'; \quad t = \gamma_V(t' + Vx'/c^2).$$

### 1.5 Transformation of the velocities

The differential form of the Lorentz transformation is:

$$dx' = \gamma_v(dx - Vdt)$$

$$dy' = dy$$

$$dz' = dz$$

$$dt' = \gamma_v(dt - Vdx/c^2).$$

Let  $\mathbf{U}$  the velocity of an event in S:

$$\mathbf{U} = (dx/dt, dy/dt, dz/dt) = (U_x, U_y, U_z).$$

In S', this velocity becomes  $\mathbf{U}'$ :

$$\mathbf{U}' = (dx'/dt', dy'/dt', dz'/dt') = (U'_x, U'_y, U'_z).$$

How do they relate to each other?

$$U'_x = \frac{dx'}{dt'} = \frac{\gamma_v(dx/dt - V)}{\gamma_v(1 - \frac{V}{c^2} \frac{dx}{dt})} = \frac{U_x - V}{1 - \frac{V}{c^2} U_x}$$

$$U'_y = \frac{dy'}{dt'} = \frac{dy/dt}{\gamma_v(1 - \frac{V}{c^2} \frac{dx}{dt})} = \frac{U_y}{\gamma_v(1 - \frac{V}{c^2} U_x)}$$

And similarly for  $U'_z$ . Note that when  $V/c \rightarrow 0$ , we recover the Galilean transformation of velocities .

Summary: Transformation of velocities:

$$\boxed{U'_x = \frac{U_x - V}{1 - \frac{V}{c^2} U_x}; \quad U'_y = \frac{U_y}{\gamma_v(1 - \frac{V}{c^2} U_x)}; \quad U'_z = \frac{U_z}{\gamma_v(1 - \frac{V}{c^2} U_x)}}$$

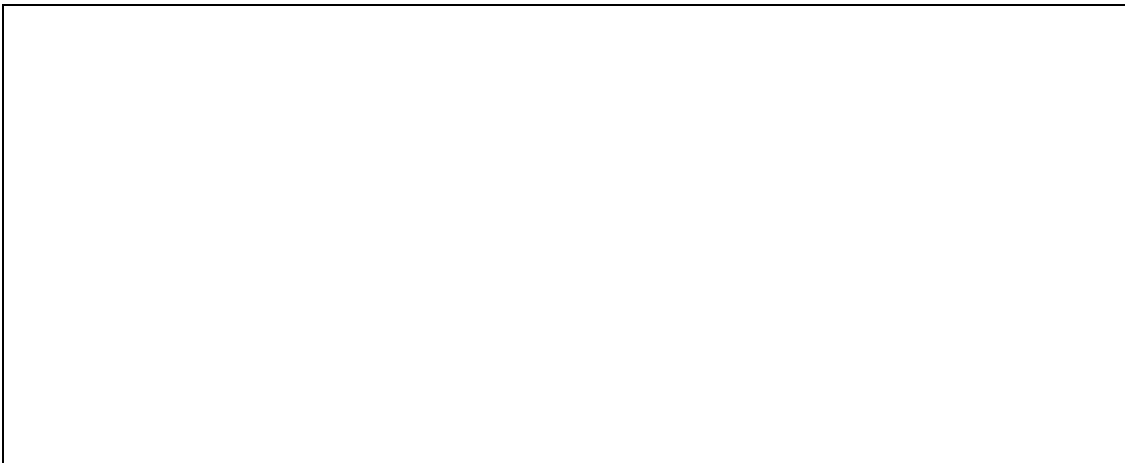
Inverse transformation:

$$\boxed{U_x = \frac{U'_x + V}{1 + \frac{V}{c^2} U'_x}; \quad U_y = \frac{U'_y}{\gamma_v(1 + \frac{V}{c^2} U'_x)}; \quad U_z = \frac{U'_z}{\gamma_v(1 + \frac{V}{c^2} U'_x)}}$$

Example: An observer in S sends a light beam in the direction  $x$ . What is the velocity of light in  $S'$ ?



Example: A particle moves along the direction  $x'$  with a speed  $c/2$  as measured in  $S'$ . The observer in  $S'$  in turn moves with a velocity  $c/2$  as measured in S along the direction  $x$ . What is the velocity of the particle as measured in S?

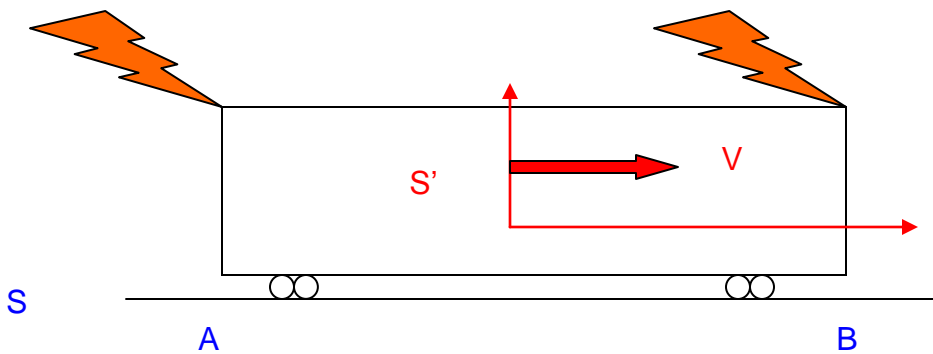


## 1.6 Consequences of the Lorentz transformations

### i) Simultaneity

If two events in  $S$  are simultaneous (i.e.  $t_1 = t_2$ ), they are not necessarily so in  $S'$  (i.e.  $t'_1 \neq t'_2$ ).

Example: The two ends of a train ( $S'$  frame) that moves at velocity  $V$  with respect to the ground ( $S$  frame) are hit by lightning bolts. The hits are seen as simultaneous in  $S$ . The observer in  $S'$ , located in the middle of the train, will see the front bolt B BEFORE the back one. This is true since the speed of light is the same in all inertial frames and the light from the front travels a shorter distance in  $S'$ .



## ii) Length contraction

Consider the act of measuring the length of a ruler. Suppose that the ruler is at rest in  $S'$ .

$$\text{« proper » length} = L_o = x'_2 - x'_1.$$

By Lorentz, let's go to a frame  $S$  in which the ruler has a velocity  $V$  along the  $x$  direction:

$$\begin{aligned} L_o = x'_2 - x'_1 &= \gamma_V [x_2 - Vt_2 - (x_1 - Vt_1)] \\ &= \gamma_V [L - V(t_2 - t_1)] \end{aligned}$$

where  $L = x_2 - x_1$ . The measure of the ruler length in  $S$  must be done in noting the locations of its two ends simultaneously ( $t_1 = t_2$ ). Then,  $L$  is the length of the moving ruler and:

$$L = \frac{L_o}{\gamma_V} = L_o \sqrt{1 - V^2 / c^2}.$$

We see that a moving ruler is always shorter than when it is at rest.

In other words, "jogging makes you look thinner".

## iii) Time dilation

Consider a time interval defined by a clock at rest in  $S'$ :

$$\text{Proper time} = T_o = t'_2 - t'_1.$$

Let's go to  $S$  in which the clock moves at velocity  $V$ . The period of the clock in  $S$  is:

$$T = t_2 - t_1 = \gamma_V (t'_2 - t'_1 + V(x'_2 - x'_1) / c^2) = \gamma_V T_o.$$

The last equality comes from the fact that, for a clock at rest,  $x'_1 = x'_2$ . Hence:

$$T = \gamma_V T_o = \frac{T_o}{\sqrt{1 - V^2 / c^2}}.$$

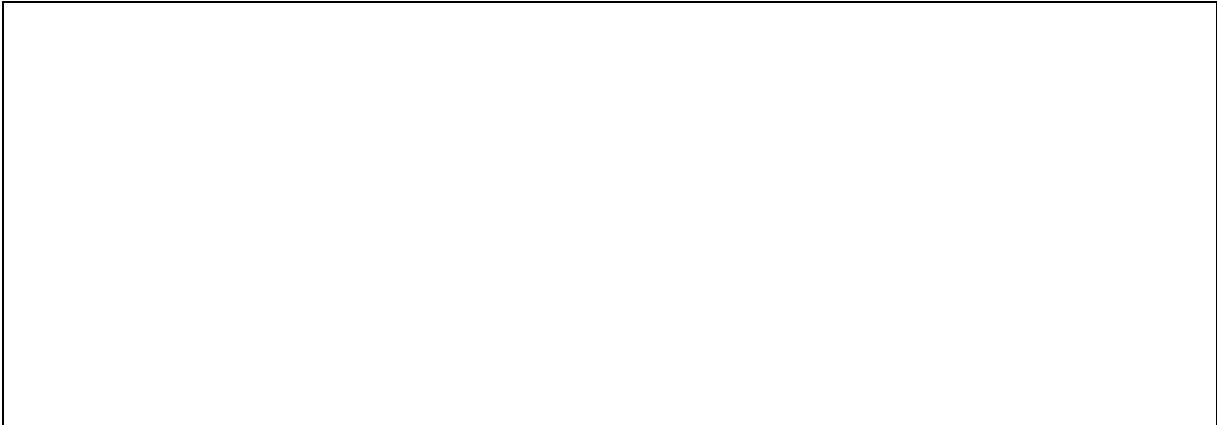
The period of a moving clock is larger than when it is at rest.

In other words, « Travellers stay young ».

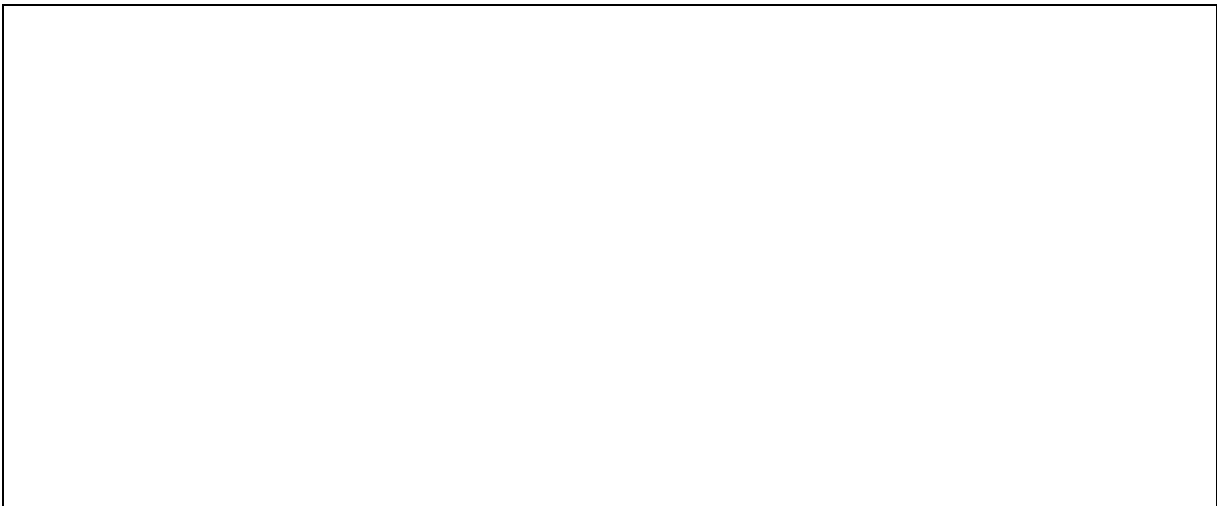
Example: Muon disintegration.

A muon is an elementary particle that may be produced in the high atmosphere when cosmic rays interact with the atmospheric gas. The muon life time is  $1.52 \mu\text{s}$  (this is the time it takes for half the population of an ensemble of muons AT REST to disintegrate). Suppose that the velocity of the muons as measured by an observer on the ground is  $V=0.98 c$ .

a) From the point of view of an observer on the ground (frame S):



b) From the point of view of the muon itself (frame S'):



### Example: The twin paradox

One can consider the clock as a biological being. Twin Castor stays on Earth while his brother Pollux is sent on a spacecraft at constant speed  $V$  for a  $T_{\text{pol}}$  (as measured by Pollux himself). Pollux then changes direction and comes back towards Earth at the same constant speed  $V$ . The total traveling time (proper time since Pollux is at rest in his own frame) is  $2 T_{\text{pol}}$ . But the travel time measured by his brother Castor is  $T_{\text{cas}} = \gamma_V 2T_{\text{pol}} > 2 T_{\text{pol}}$ . When the two twins reunite, they realize that Castor is older than Pollux...

That is fine, but here comes the paradox: one can argue that the two frames are equivalent, so that for Pollux, it is Castor who traveled back and forth at the same speed  $V$ . Each twin then concludes that the other is older when they reunite. Who is right?

The paradox can be resolved when it is realized that the two frames are NOT equivalent. Pollux was subjected to an acceleration when he turned back. Part of the trip lies outside the realm of special relativity, since Pollux's frame is not always inertial. This phenomenon was confirmed directly by experience, see J.C. Hafele et R.E. Keating (1972), *Science* **177**, 166.

### 1.7 Interval

By construction of the Lorentz transformation, we have:

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2 \equiv s^2.$$

For the case of a beam of light,  $s^2 = 0$ . We call  $s^2$  the INTERVAL between  $(0,0,0,0)$  and  $(x,y,z,t)$  and one sees that its numerical value is INVARIANT (i.e. the same in all inertial frames). One can write  $ct \equiv x_0$ ;  $x \equiv x_1$ ;  $y \equiv x_2$ ;  $z \equiv x_3$ . The invariance of the interval implies:

$$\sum_{\mu, \nu=0}^3 x_{\mu} x_{\nu} g_{\mu\nu} = s^2 = \sum_{\mu, \nu=0}^3 x'_{\mu} x'_{\nu} g_{\mu\nu}.$$

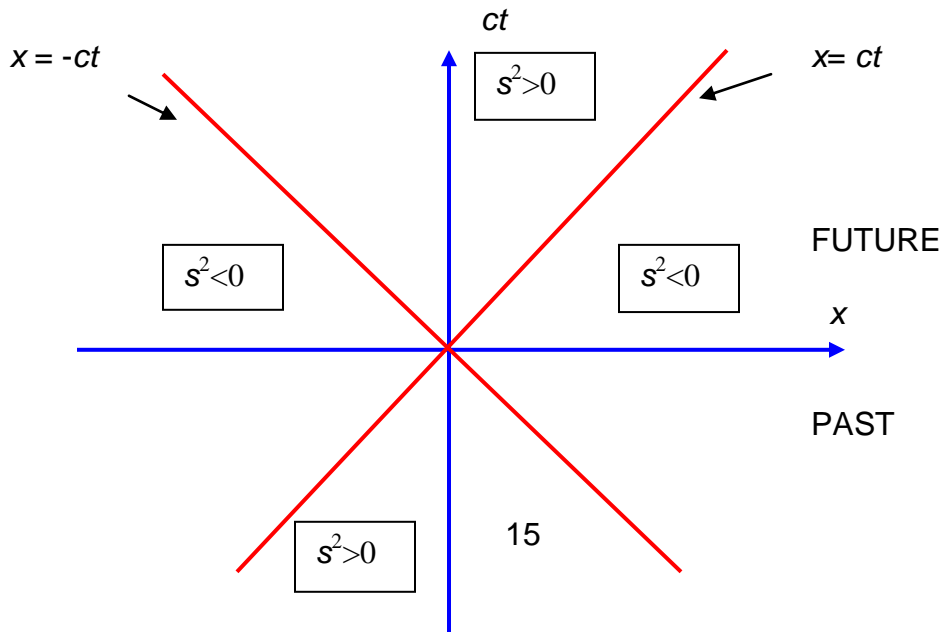
Here, we have introduced a diagonal matrix  $\mathbf{g}$  called the **metric**:

$$\mathbf{g} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Thus, the Lorentz transformations keep the “length” of the 4-dimensional vector position-time unchanged. This corresponds to a “rotation” of the 4-vector position-time in space-time.

Special cases:

- $s^2 = 0$ : The events  $(0,0,0,0)$  and  $(x,y,z,t)$  are connected by a light beam. One says that the interval is of the « light kind ».
- $s^2 < 0$ : The events  $(0,0,0,0)$  and  $(x,y,z,t)$  cannot be causally related. They can be simultaneous but take place at different locations. The interval is said of the « space kind ».
- $s^2 > 0$ : The events  $(0,0,0,0)$  and  $(x,y,z,t)$  can occur at the same position but cannot be simultaneous. There can be a causal relation between the two events. The interval is said to be of the « time kind ».



Examples of the use of the concept of interval:

- Length contraction revisited

One measures the length of a moving ruler. The differential interval between the two ends of the ruler is invariant:  $\Delta s^2 = -(\Delta x)^2 + c^2(\Delta t)^2$ .

In S (where the ruler moves):  $\Delta x = L, \Delta t = 0 \Rightarrow \Delta s^2 = -L^2$ .

In S' (where the ruler is at rest):  $\Delta x' = L_0, \Delta t' = -V\Delta x'/c^2 \Rightarrow \Delta s^2 = -L_0^2(1 - V^2/c^2)$ .

By invariance of the interval:  $L = L_0\sqrt{1 - V^2/c^2}$ .

- Time dilation revisited:

One measures the period of a moving clock.

In S' (clock at rest):  $\Delta x' = 0; \Delta t' = T_0 \Rightarrow \Delta s^2 = T_0^2 c^2$ .

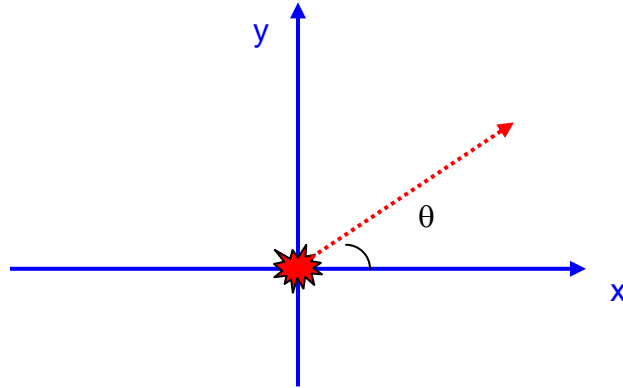
In S (where the clock is moving):  $\Delta t = T; \Delta x = V\Delta t \Rightarrow \Delta s^2 = T^2 c^2 - V^2 T^2$ .

By invariance of the interval:  $T = T_0/\sqrt{1 - V^2/c^2}$ .

### 1.8 Relativistic Doppler effect

The Doppler effect consists in the existence of a shift in the frequency of a wave due to the relative motion between the source and the observer.

Let a source of light at rest in S emitting at a frequency  $f$  and at a wavelength  $\lambda$  in a direction  $\theta$  with respect to the direction  $x$ :



We introduce the wave-number vector ( $\text{m}^{-1}$ ) whose magnitude is  $k=|\mathbf{k}| = 2\pi/\lambda$  and an orientation given by the direction of the light beam. We also introduce the angular frequency (in rad/s)

$$\omega = 2 \pi f.$$

We have:  $c = \lambda f$  or  $\omega = k c$ , which is:

$$\frac{\omega^2}{c^2} - k^2 = 0.$$

In another frame  $S'$ ,  $\mathbf{k}$  and  $\omega$  are not the same any more. They are denoted  $\mathbf{k}'$  et  $\omega'$ . But we will always have  $\omega' = k' c$ , which is:

$$\frac{\omega'^2}{c^2} - k'^2 = 0.$$

By analogy with the invariance property of the interval, one can define the following 4-vector:

$$K_0 \equiv \omega/c; \quad K_1 \equiv k_x; \quad K_2 \equiv k_y; \quad K_3 \equiv k_z$$

with the property that:

$$\sum_{\mu, \nu=0}^3 K_\mu K_\nu g_{\mu\nu} = 0.$$

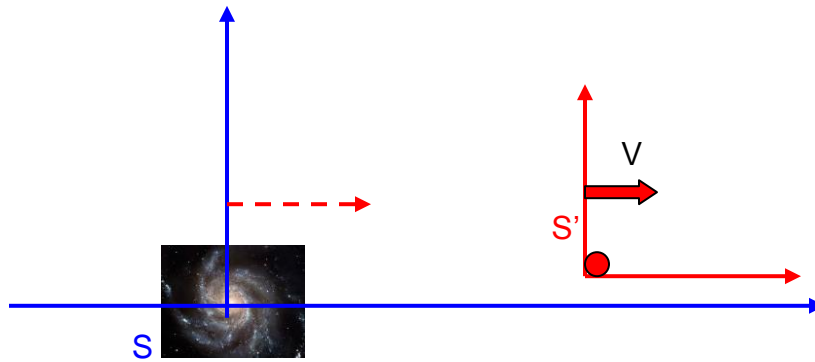
The magnitude of the 4-vector wave-number  $K_\mu$  is thus invariant in a Lorentz transformation. Each component transforms according to Lorentz. In particular:

$$\omega' = \frac{\omega - V k_x}{\sqrt{1 - V^2/c^2}}.$$

This relation is the analog of the transformation  $t' = \gamma_V(t - Vx/c^2)$  with the substitution  $x \rightarrow k_x$ ;  $ct \rightarrow \omega/c$ . Since  $k_x = k \cos \theta = (\omega/c) \cos \theta$ , one recovers the relativistic Doppler effect that relate the frequency in S and S':

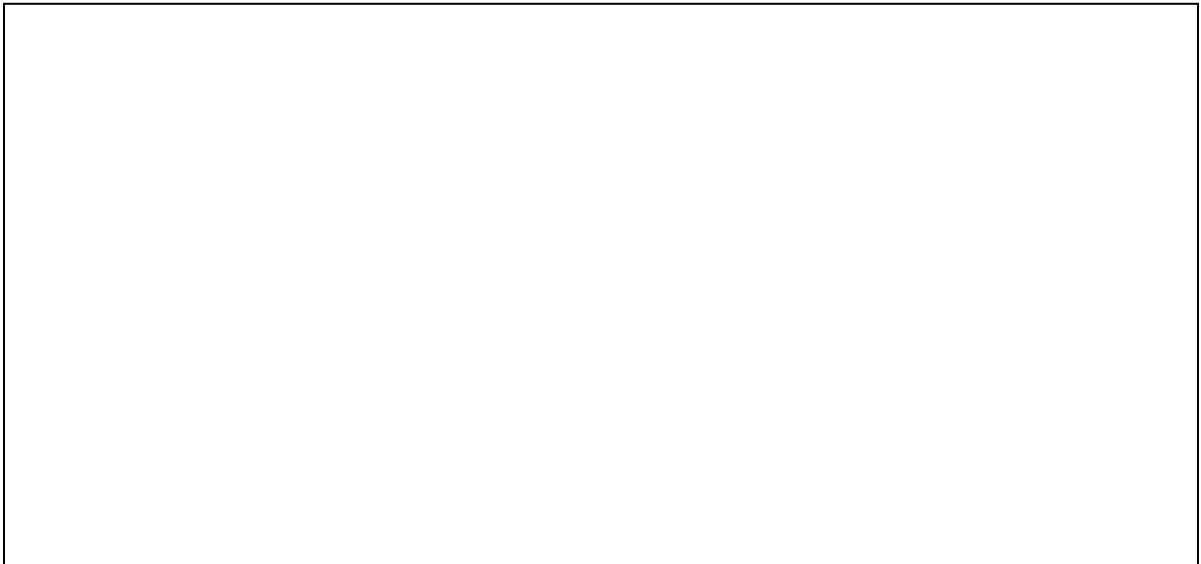
$$f' = f \frac{(1 - \frac{V}{c} \cos \theta)}{\sqrt{1 - V^2/c^2}}.$$

Example: Red shift of far away galaxies (those get away from the Earth).



GALAXY

EARTH



## 1.2 RELATIVISTIC DYNAMICS

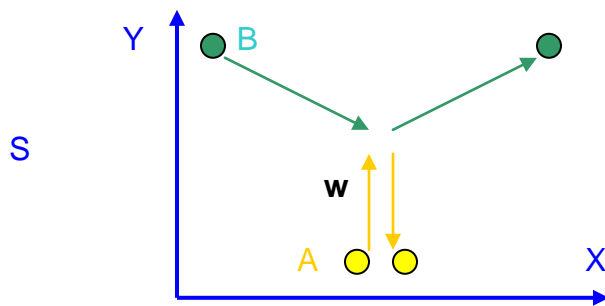
*“The hardest thing in the world to understand is income taxes.” I replied: “There is one thing more difficult, and that is your theory of relativity”. “Oh, no,” he replied, “that is easy.”*

*Einstein to Leo Mattersdorf, tax accountant*

### 1.1 Momentum

We consider an elastic collision between two identical balls of mass  $m$ :

In  $S$ , the ball A is launched along the  $y$  axis with a speed  $w$ .

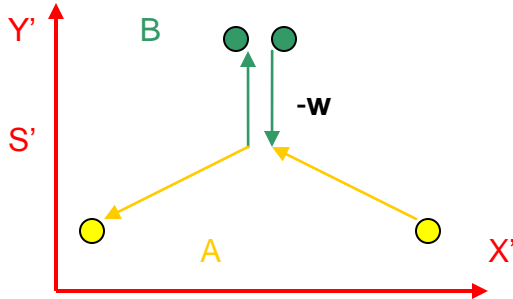


Before collision:  $U_{Ax} = 0$ ;  $U_{Ay} = w$ ;  $p_{Ay} = mw$ .

After collision:  $U_{Ax} = 0$ ;  $U_{Ay} = -w$ ;  $p_{Ay} = -mw$ .

The momentum change of A in the collision is  $\Delta p_{Ay} = -2mw$ .

Suppose S' has velocity  $V=U_{Bx}$ . Also, in S', the ball B is thrown in the  $-y'$  direction with a speed  $U'_{By} = -w$ .



We have from the velocity transformation rules:

$$U_x = (U'_x + V) / (1 + VU'_x / c^2); \quad U_y = \sqrt{1 - V^2 / c^2} U'_y / (1 + VU'_x / c^2).$$

Before collision:  $U'_{Bx} = 0; \quad U'_{By} = -w \Rightarrow U_{Bx} = V; \quad U_{By} = -w\sqrt{1 - V^2 / c^2}.$

After collision:  $U'_{Bx} = 0; \quad U'_{By} = w \Rightarrow U_{Bx} = V; \quad U_{By} = w\sqrt{1 - V^2 / c^2}.$

One concludes that the momentum change of ball B in the y direction as measured in the S frame is

$$\Delta p_{By} = 2mw\sqrt{1 - V^2 / c^2} \neq -\Delta p_{Ay} (!)$$

Momentum is not conserved! But, that is awful!

We must do something. Einstein has solved the situation by redefining momentum, in such a way that it is conserved in a collision. Let's generalize the definition of momentum of a mass moving with a velocity  $\mathbf{u}$  as:

$$\mathbf{p} = A(u)m\mathbf{u}$$

with  $A \rightarrow 1$  when  $u/c \ll 1$  (in order to recover the standard nonrelativistic definition of momentum). One finds:  $A = 1 / \sqrt{1 - u^2 / c^2}$ . Hence the new definition of the relativistic momentum:

$$\mathbf{p} = \frac{m\mathbf{u}}{\sqrt{1 - u^2 / c^2}} = \gamma_u m\mathbf{u}$$

where  $\gamma_u \equiv (1 - u^2 / c^2)^{-1/2}$ .

Note: some authors interpret this relation by writing  $\mathbf{p} = m(u) \mathbf{u}$  and saying that the mass  $m(u) = \gamma_u m(0)$  depends on the speed. We will not do this here. For us,  $m$  will always denote the mass of a particle at rest.

We now verify that this definition makes sense and that the momentum is conserved in the collision we studied previously.

Ball A in S:

$$\text{Before collision: } p_{Ay} = mw / \sqrt{1 - w^2/c^2}$$

$$\text{After collision: } p_{Ay} = -mw / \sqrt{1 - w^2/c^2}$$

$$\text{Momentum change of ball A: } \Delta p_{Ay} = -2mw / \sqrt{1 - w^2/c^2}.$$

Ball B in S:

Before collision:

$$\begin{aligned} p_{By} &= mU_{By} / \sqrt{1 - (U_{By}^2 + U_{Bx}^2)/c^2} = \frac{-mw\sqrt{1 - V^2/c^2}}{(1 - (w/c)^2(1 - V^2/c^2) - V^2/c^2)^{1/2}} \\ &= -\frac{mw\sqrt{1 - V^2/c^2}}{\sqrt{1 - V^2/c^2} \sqrt{1 - w^2/c^2}} = -\frac{mw}{\sqrt{1 - w^2/c^2}}. \end{aligned}$$

$$\text{After collision: } p_{By} = \frac{mw}{\sqrt{1 - w^2/c^2}}.$$

$$\text{One concludes that the momentum change of B is: } \Delta p_{By} = \frac{2mw}{\sqrt{1 - w^2/c^2}}.$$

$$\text{The momentum is then conserved: } \Delta p_{By} + \Delta p_{Ay} = 0.$$

## 2.2 Energy

i) Definition:

One first writes the following algebraic identity:  $\gamma_u^2(c^2 - u^2) = c^2$ .

The differential form of this relation is:

$$(c^2 - u^2)\gamma_u d\gamma_u - \gamma_u^2 \mathbf{u} \cdot d\mathbf{u} = 0 \Rightarrow \gamma_u \mathbf{u} \cdot d\mathbf{u} = c^2 d\gamma_u - u^2 d\gamma_u.$$

Newton's second law gives:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\gamma_u m \mathbf{u}) = m \mathbf{u} \frac{d\gamma_u}{dt} + \gamma_u m \frac{d\mathbf{u}}{dt}.$$

The work done by the force over a displacement  $d\mathbf{x}$  is:

$$dW = \mathbf{F} \cdot d\mathbf{x} = m \mathbf{u} \cdot d\mathbf{x} \frac{d\gamma_u}{dt} + \gamma_u m \frac{d\mathbf{u}}{dt} \cdot d\mathbf{x} = m u^2 d\gamma_u + \gamma_u m \mathbf{u} \cdot d\mathbf{u}.$$

Using the above identity:  $dW = m c^2 d\gamma_u$ .

Integrating from the state of rest, one gets the kinetic energy  $K$ :

$$K = \int_0^u m c^2 d\gamma_u = m c^2 (\gamma_u - 1) = m c^2 \left( \frac{1}{\sqrt{1 - u^2/c^2}} - 1 \right).$$

One recovers the standard non-relativistic case  $u \ll c$ , by expanding the square root:

$$K = m c^2 \left( 1 + \frac{u^2}{2c^2} + \dots - 1 \right) \cong \frac{1}{2} m u^2 + \dots$$

One can thus interpret  $K + m c^2 = \gamma_u m c^2$  as a total mechanical energy  $E$  that includes a kinetic contribution and a part from the mass at the state of rest  $m c^2$ :

$$E = \gamma_u m c^2 = \frac{m c^2}{\sqrt{1 - u^2/c^2}}.$$

This is the famous Einstein relation, that everyone knows, even my mother-in-law.

ii) Relation E-p:

Please, recall that:  $\mathbf{p} = \gamma_u m \mathbf{u} = \gamma_u m c^2 \frac{\mathbf{u}}{c^2} = E \frac{\mathbf{u}}{c^2}$ .

On the other hand,  $E^2 = \gamma_u^2 m^2 c^4 = \frac{m^2 c^4}{1 - u^2/c^2} \Rightarrow E^2 - E^2 \frac{u^2}{c^2} = m^2 c^4$ .

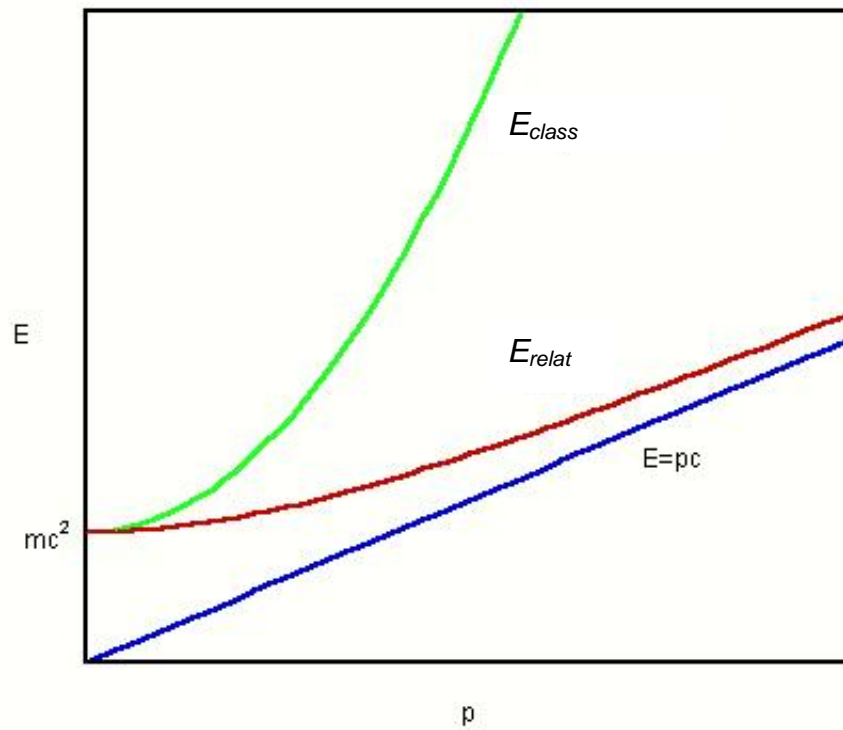
Hence:  $E^2 - p^2 c^2 = m^2 c^4$ . Thus:

$$\boxed{E^2 = p^2 c^2 + m^2 c^4.}$$

This relation is the relativistic analogue of the classical formula:

$$E = mc^2 + \frac{1}{2} m u^2 = mc^2 + \frac{p^2}{2m}.$$

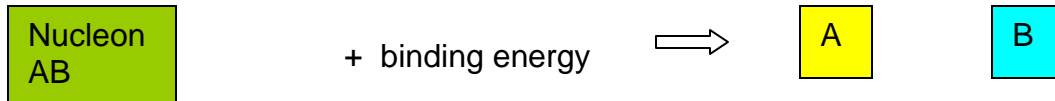
For light, one has  $m=0$  and  $E=pc$  which is a well known result also obtained from classical electromagnetism.



iii) Binding energy:

Energy and mass are equivalent. This is the principle behind the amount of energy freed in a nuclear fusion or fission reaction.

Example: binding of nucleons:



(The inverse process would be a nuclear fusion reaction.)

From the conservation of TOTAL energy (including the rest mass contribution):

$$M_{AB} c^2 + \text{binding energy} = m_A c^2 + m_B c^2.$$

One concludes that  $M_{AB} \leq m_A + m_B$ .

For example, let's consider the deuteron (proton + neutron):

$$\begin{aligned} m_p &= 1.007276 \text{ ua} \\ m_n &= 1.008665 \text{ ua} \\ m_d &= 2.013553 \text{ ua} \end{aligned}$$

(1 ua =  $1.66 \times 10^{-27}$  kg). Find  $\Delta m = m_p + m_n - m_d$  and the equivalent energy.

### 2.3 Energy-momentum 4-vector:

One recalls  $E^2/c^2 - p^2 = m^2c^2$  from the E-p relationship. But the quantity on the right-hand side is invariant and does not depend on the choice of frame, since everybody will agree on its numerical value. Indeed, everybody agrees on the numerical value of the mass of a given body in the state of rest.

We introduce:  $P_0 \equiv E/c$ ;  $P_1 \equiv p_x$ ;  $P_2 \equiv p_y$ ;  $P_3 \equiv p_z$ .

Thus:

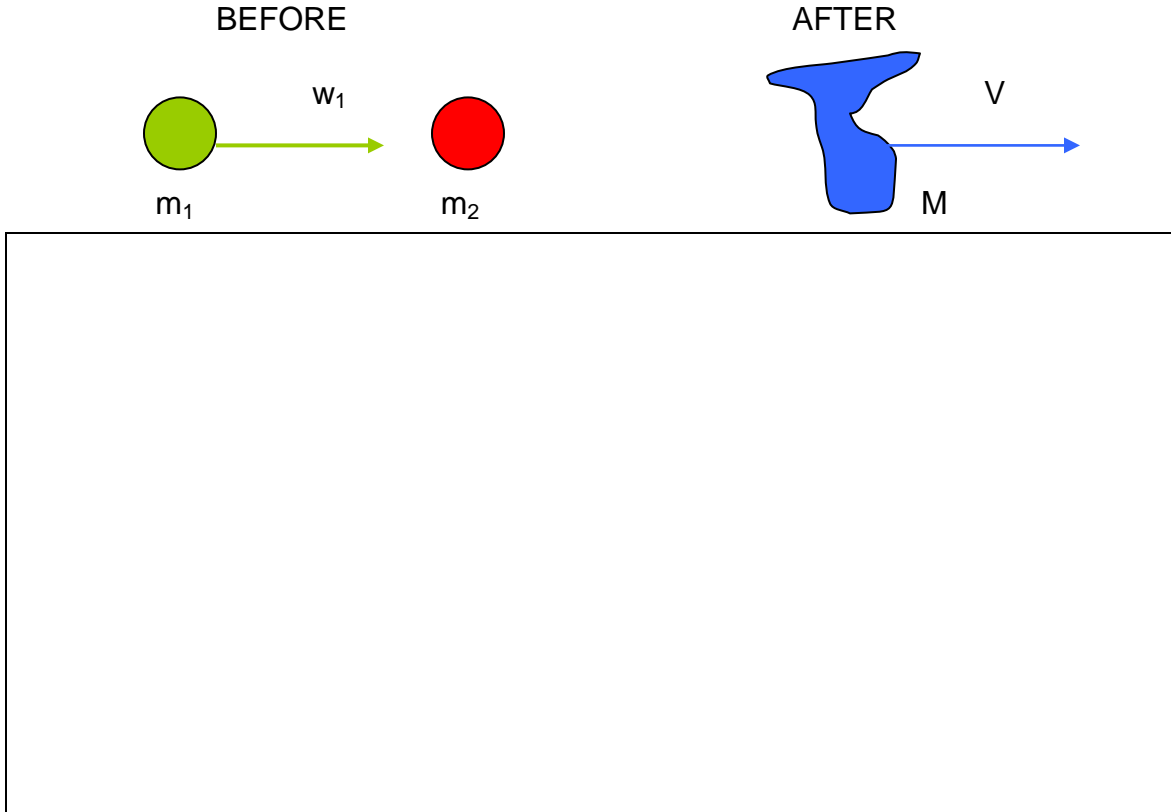
$$\sum_{\mu, \nu=0}^3 P_\mu P_\nu g_{\mu\nu} = m^2 c^2.$$

Consequently, the  $P_\mu$  define another 4-vector, whose magnitude is invariant in a Lorentz transformation. Thus individual components of this 4-vector (called energy-momentum) will transform under Lorentz in analogy with the components of the 4-vector position in space-time. Substituting  $x, y, z$  by  $p_x, p_y, p_z$  and  $ct$  by  $E/c$  in the Lorentz transformations, one gets:

$$p'_x = \frac{p_x - EV/c^2}{\sqrt{1 - V^2/c^2}}; \quad p'_y = p_y; \quad p'_z = p_z; \quad E' = \frac{E - Vp_x}{\sqrt{1 - V^2/c^2}}.$$

So, in a collision, the 4 components of the 4-vector  $(E/c, \mathbf{p})$  are individually conserved.

Example: A particle of mass  $m_1$  with speed  $w_1$  is subjected to a completely inelastic collision with another particle of mass  $m_2$ , which is at rest. Find the velocity  $V$  of the resulting composite of mass  $M$ .



One can also say that  $V$  is the velocity of the center of mass. In that frame, the total momentum is simply 0:

$$p'_x = 0 = \frac{p_x - EV/c^2}{\sqrt{1 - V^2/c^2}} \Rightarrow V = \frac{p_x c^2}{E} = \frac{\gamma_1 m_1 w_1}{\gamma_1 m_1 + m_2}.$$

which gives the same answer.

### 1.3 GENERAL RELATIVITY

“Asked in 1919 whether it was true that only three people in the world understood the theory of general relativity, Eddington allegedly replied: 'Who's the third?'”  
— Arthur Stanley Eddington

- **Covariance principle:**

All laws of physics have the same form in all frames of reference.

An accelerated system implies the presence of inertial forces that can be related to a curvature of space-time.

- **Equivalence principle:**

Inertial mass = gravitational mass

An accelerated system is thus equivalent to a system in a gravitational field.

- **Geodesic**

The local deformation of the geometry of space-time is described by a 4 X 4 symmetric matrix (or « tensor ») called metric  $g_{\mu\nu}$ . The metric defines how distances are computed in space-time:

$$ds^2 = \sum_{\mu,\nu} g_{\mu\nu} dx^\mu dx^\nu .$$

The first law of Newton is generalized by saying that the trajectory of a particle that is not subjected to any force (except gravity) becomes a **geodesic**. The effect of gravity is to create a local curvature in space-time due to the presence of masses in the neighborhood. The motion of an object in this gravity field is not due to a Newtonian force acting at a distance, but simply to follow the local geodesic curve. The equation of motion of the particle becomes

$$\ddot{x}^\alpha + \sum_{\beta\gamma} \Gamma_{\beta\gamma}^\alpha \dot{x}^\beta \dot{x}^\gamma = 0$$

where  $x^\alpha$  are the components of the 4-vector position in space-time (technically, the so-called “contravariant” components), the dot indicates a derivative with respect to proper time and the factors  $\Gamma_{\beta\gamma}^\alpha$  (called Christoffel symbols) are functions of the first derivatives of  $g_{\mu\nu}$ .

- **The famous Einstein equation**

If you want to impress your (future?) mother in law, tell her that the modern theory of gravitation is described by:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

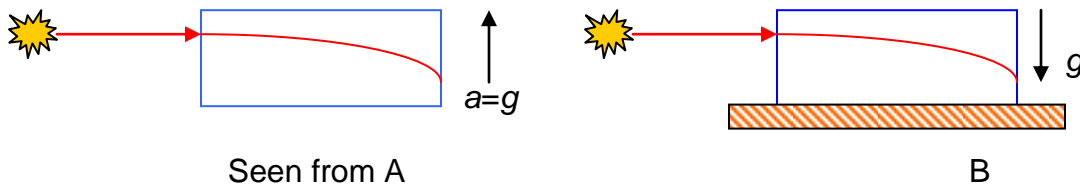
where  $R_{\mu\nu}$  is the Ricci tensor,  $g_{\mu\nu}$  is the metric tensor and  $R$  is the curvature scalar. These terms describe the geometrical structure of space-time.

$\lambda$  is the cosmological constant,  $G$  is the gravitation constant and  $T_{\mu\nu}$  is the energy-momentum tensor, that defines the content and transport of energy and momentum in the universe. We do not have time to explain this in details, but I wrote it down for your general culture.

- **Consequences:**

1) Deviation of light by gravity:

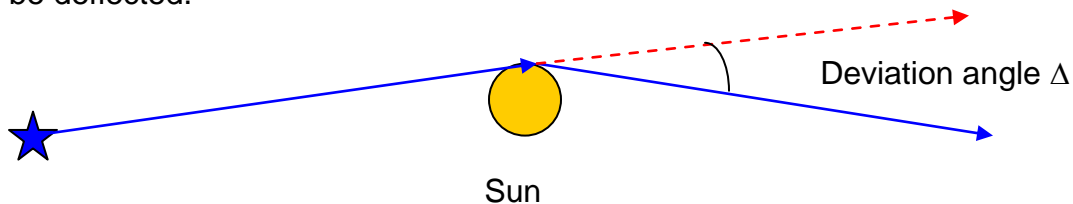
A is in an accelerated lab; B is an external gravity field.



For an observer in A, the light has a curved trajectory.

For B, according to the two principles above, light must also follow a curved trajectory.

So, a light beam emitted from a star that passes close to the Sun's surface will be deflected.



Einstein has calculated (1915)  $\Delta = 1.75''$ .

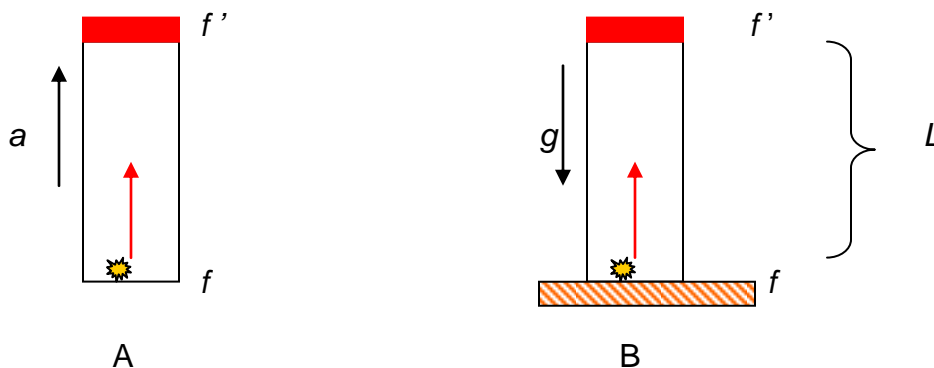
Addington has then confirmed this prediction during the solar eclipse of 1919. Einstein became an international « star ».

Since, many observations have confirmed this phenomenon (deviation of a radar beam going from Earth to a planet, deviation of radio waves coming from quasars and passing close to the Sun's surface, etc.)



## 2) Doppler effect:

Consider the emission of a light beam in an accelerated frame (A) and in a frame subjected to a gravitational field (B). The beam is detected at the top of the frame.



For A: the time of flight is  $t = L/c$ .

The change of velocity of the detector during that time is  $\Delta v = a t = aL/c$ .

Using the formula for the Doppler effect:  $\Delta f/f \cong -\Delta v/c = -aL/c^2$ .

For B: According to the principle of covariance, the Doppler effect must still exist:

$$\Delta f/f \cong -gL/c^2 = -\phi/c^2$$

where  $\phi$  is the gravitational potential of the detector.

That effect has been experimentally confirmed in 1960 by Mössbauer effect.

### 3) Precession of the orbit of Mercury:

After due consideration of the perturbations caused by the presence of Venus, the Earth and Jupiter, there remains an unexplained precession rate of 43 " per century in the orbit of Mercury. General relativity gives an explanation for this residual precession rate.

### 4) Radiation of gravitational waves:

The theory predicts that gravitational waves exist and propagate at velocity  $c$ . They correspond to perturbations in the metric of space-time. Many researchers are currently busy trying to detect these waves.

An indirect evidence came from the measurement of the slowing down of the period of the orbit of a companion rotating around a pulsar. The slowing down rate is  $76 \mu\text{s}/\text{year}$ .

### 5) Black holes:

Singularities in space-time may exist. Even light can not escape from the gravity field of such "object" when it is emitted at a distance closer to the object than the so-called **Schwartzchild radius**  $R_s=2GM/c^2$  where  $M$  is the mass of the singularity and  $G$  the constant of gravitation.

6) **Cosmology**: is the Universe closed or open? Nobody knows, although we have hints that it is open and that its expansion rate is increasing.