(A)

MAT 2384 3X
DIFFERENTIAL EQUATIONS AND NUMERICAL METHODS
TEST #1

June 1, 2016

Instructor: Dr. Steve Desjardins

Duration: 80 minutes

Name: Solutions

Student Number:________________

Instructions:

- Print your name and student number on this page.
- Verify that your copy of the exam has all 5 pages.
- You must answer all questions.
- Write your answers in the spaces below the questions. You may use the backs of the pages if necessary.
- Basic scientific calculators only - graphing and/or programmable calculators are NOT permitted.
- Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam, academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam. By signing below, you acknowledge that you have ensured that you are complying with the above statement.

Signature:________________

(A)

Question 1 (5 marks) Find the general solutions of the differential equations:

(a) $\cos y\, y' = \dfrac{4x}{1+x^2}$ DE is separable

$$\cos y \, dy = \frac{4x}{1+x^2}\, dx$$

so $$\int \cos y \, dy = \int \frac{4x}{1+x^2}\, dx + C$$

$$\sin y = 2\ln(1+x^2) + C$$

$$\therefore \boxed{y = \arcsin\left(2\ln(1+x^2) + C\right)}$$

(b) $y' + \dfrac{2}{x} y = x^2 + 2$ DE is linear with $f(x) = \frac{2}{x}$, $r(x) = x^2 + 2$

$$\mu(x) = e^{\int f(x)dx} = e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^2$$

then $$y = \frac{1}{\mu(x)}\left[\int \mu(x) r(x) dx + C\right]$$

$$= x^{-2}\left(\int (x^2)(x^2+2)\, dx + C\right)$$

$$= x^{-2}\left(\int (x^4 + 2x^2)\, dx + C\right)$$

$$= x^{-2}\left(\frac{1}{5}x^5 + \frac{2}{3}x^3 + C\right)$$

$$= \boxed{\frac{1}{5}x^3 + \frac{2}{3}x + Cx^{-2}}$$

Question 2 (5 marks) Find the general solution:

$$(24xy^2 - 6y^4 + 3e^y)\,dx + (12x^2y - 8xy^3 + xe^y)\,dy = 0$$ (not separable)

$M(x,y) = 24xy^2 - 6y^4 + 3e^y \Rightarrow M_y = 48xy - 24y^3 + 3e^y$

$N(x,y) = 12x^2y - 8xy^3 + xe^y \Rightarrow N_x = 24xy - 8y^3 + e^y$

$M_y \neq N_x$, DE not exact

$$\frac{M_y - N_x}{N} = \frac{24xy - 16y^3 + 2e^y}{12x^2y - 8xy^3 + xe^y} = \frac{2}{x} \text{ (a function of } x \text{ only)}$$

then $\mu(x) = e^{\int \frac{2}{x}dx} = x^2$ and the DE becomes

$$(24x^3y^2 - 6x^2y^4 + 3x^2e^y)dx + (12x^4y - 8x^3y^3 + x^3e^y)dy = 0$$

$M^*(x,y) = 24x^3y^2 - 6x^2y^4 + 3x^2e^y \Rightarrow M^*_y = 48x^3y - 24x^2y^3 + 3x^2e^y$

$N^*(x,y) = 12x^4y - 8x^3y^3 + x^3e^y \Rightarrow N^*_x = 48x^3y - 24x^2y^3 + 3x^2e^y$

$M^*_y = N^*_x$, so DE now exact

$$F(x,y) = \int M^*(x,y)dx + g(y) = \int(24x^3y^2 - 6x^2y^4 + 3x^2e^y)dx + g(y)$$
$$= 6x^4y^2 - 2x^3y^4 + x^3e^y + g(y)$$

then $\dfrac{\partial F}{\partial y} = 12x^4y - 8x^3y^3 + x^3e^y + g'(y)$

$= N^*(x,y) = 12x^4y - 8x^3y^3 + x^3e^y$

$\Rightarrow g'(y) = 0 \Rightarrow$ take $g(y) = 0$

so $F(x,y) = 6x^4y^2 - 2x^3y^4 + x^3e^y$

and $\therefore$ the general solution is $\boxed{6x^4y^2 - 2x^3y^4 + x^3e^y = C}$

A

Question 3 (5 marks) Solve the initial value problems:

(a) $y'' - \pi^2 y = 0, \quad y(0) = 1, \quad y'(0) = \pi$

the characteristic equation is $\lambda^2 - \pi^2 = (\lambda + \pi)(\lambda - \pi) = 0$

so $\lambda_{1,2} = \pm\pi$ and the general solution

is $y(x) = c_1 e^{\pi x} + c_2 e^{-\pi x}$

$y(0) = 1 \Rightarrow 1 = c_1 e^0 + c_2 e^0 \Rightarrow c_1 + c_2 = 0$

$y'(x) = \pi c_1 e^{\pi x} - \pi c_2 e^{-\pi x}$

$y'(0) = \pi \Rightarrow \pi = \pi c_1 e^0 - \pi c_2 e^0 \Rightarrow c_1 - c_2 = 1$

so $c_1 = 1$ and $c_2 = 0$

$\therefore$ the unique solution is $\boxed{y(x) = e^{\pi x}}$

(b) $y'' - 8y' + 20y = 0, \quad y(0) = 2 \quad y'(0) = 14$

Char. eq. is $\lambda^2 - 8\lambda + 20 = 0$

so roots are $\lambda_{1,2} = \dfrac{8 \pm \sqrt{(-8)^2 - 4(20)}}{2} = \dfrac{8 \pm \sqrt{-16}}{2} = 4 \pm 2i$

and the general solution is

$y(x) = c_1 e^{4x} \cos(2x) + c_2 e^{4x} \sin(2x)$

$y(0) = 2 \Rightarrow 2 = c_1 e^0 \cos(0) + c_2 e^0 \sin(0) \Rightarrow c_1 = 2$

$y'(x) = 4c_1 e^{4x} \cos(2x) - 2c_1 e^{4x} \sin(2x) + 4c_2 e^{4x} \sin(2x)$
$+ 2c_2 e^{4x} \cos(2x)$

$y'(0) = 14 \Rightarrow 14 = 4c_1 e^0 \cos(0) - 2c_1 e^0 \sin(0) + 4c_2 e^0 \sin(0) + 2c_2 e^0 \cos(0)$
$= 4c_1 + 2c_2 \Rightarrow c_2 = 3$

$\therefore$ the unique solution is $\boxed{y(x) = 2e^{4x} \cos(2x) + 3e^{4x} \sin(2x)}$

(A)

Question 4 (5 marks) Use Newton's Method to find the point of intersection of $p(x) = 4 - x^2$ and $q(x) = \ln x$ to 6 decimal places. Start with $x_0 = 1$.

let $f(x) = 4 - x^2 - \ln x$

then $f'(x) = -2x - \frac{1}{x}$

so $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$= x_n - \dfrac{4 - x_n^2 - \ln x_n}{-2x_n - \frac{1}{x_n}}$

$= x_n + \dfrac{4 - x_n^2 - \ln x_n}{2x_n + \frac{1}{x_n}}$

$= \dfrac{x_n^2 + 5 - \ln x_n}{2x_n + \frac{1}{x_n}}$

$x_0 = 1$, $x_1 = \dfrac{1 + 5 - \ln(1)}{2(1) + \frac{1}{1}} = 2$

$x_2 = 1.845\,967$

$x_3 = 1.841\,102$

$x_4 = 1.841\,097 = x_5$ $\therefore$ stop

$p(1.841\,097) = 0.610\,362$
$q(1.841\,097) = 0.610\,362$ } okay!

the point of intersection is $\boxed{(1.841097, 0.610362)}$

(B)

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Question 1 (5 marks) Find the general solutions of the differential equations:

(a) $\sec^2 y\, y' = 4e^{2x}$ DE is separable

$\sec^2 y\, dy = 4e^{2x}\, dx$

so $\int \sec^2 y\, dy = \int 4e^{2x}\, dx + C$

$\tan y = 2e^{2x} + C$

$\boxed{y = \arctan(2e^{2x} + C)}$

(b) $y' + \frac{3}{x}y = 2x + 1$ DE is linear with $f(x) = \frac{3}{x}$, $r(x) = 2x+1$

$\mu(x) = e^{\int \frac{3}{x} dx} = x^3$

$y = \frac{1}{x^3}\left[\int (x^3)(2x+1)\, dx + C \right]$

$= x^{-3}\left(\int (2x^4 + x^3)\, dx + C \right)$

$= x^{-3}\left(\frac{2}{5}x^5 + \frac{1}{4}x^4 + C \right)$

$= \boxed{\frac{2}{5}x^2 + \frac{1}{4}x + Cx^{-3}}$

Question 2 (5 marks) Find the general solution:

$$(30x^3y^3 - 4xy^2 + 4x\cos y)\,dx + (15x^4y^2 - 2x^2y - x^2\sin y)\,dy = 0$$ (not separable)

$M(x,y) = 30x^3y^3 - 4xy^2 + 4x\cos y \Rightarrow M_y = 90x^3y^2 - 8xy - 4x\sin y$

$N(x,y) = 15x^4y^2 - 2x^2y - x^2\sin y \Rightarrow N_x = 60x^3y^2 - 4xy - 2x\sin y$

$M_y \neq N_x$ DE not exact

$$\frac{M_y - N_x}{N} = \frac{30x^3y^2 - 4xy - 2x\sin y}{15x^4y^2 - 2x^2y - x^2\sin y} = \frac{2}{x} \Rightarrow \mu(x) = x^2$$

DE becomes $(30x^5y^3 - 4x^3y^2 + 4x^3\cos y)dx + (15x^6y^2 - 2x^4y - x^4\sin y)dy = 0$

$M^*(x,y) = 30x^5y^3 - 4x^3y^2 + 4x^3\cos y \Rightarrow M^*_y = 90x^5y^2 - 8x^3y - 4x^3\sin y$

$N^*(x,y) = 15x^6y^2 - 2x^4y - x^4\sin y \Rightarrow N^*_x = 90x^5y^2 - 8x^3y - 4x^3\sin y$

$M^*_y = N^*_x \Rightarrow$ DE now exact

$F(x,y) = \int N^*(x,y)\,dy + g(x) = \int (15x^6y^2 - 2x^4y - x^4\sin y)\,dy + g(x)$

$= 5x^6y^3 - x^4y^2 + x^4\cos y + g(x)$

$\frac{\partial F}{\partial x} = 30x^5y^3 - 4x^3y^2 + 4x^3\cos y + g'(x)$

$= M^*(x,y) = 30x^5y^3 - 4x^3y^2 + 4x^3\cos y$

$\Rightarrow g'(x) = 0 \Rightarrow$ take $g(x) = 0$

$F(x,y) = 5x^6y^3 - x^4y^2 + x^4\cos y$

$\therefore$ the general solution is $\boxed{5x^6y^3 - x^4y^2 + x^4\cos y = C}$

Question 3 (5 marks) Solve the initial value problems:

(a) $y'' - \pi^2 y = 0, \quad y(0) = 1, \quad y'(0) = -\pi$

general solution is $y(x) = c_1 e^{\pi x} + c_2 e^{-\pi x}$

$y(0) = 1 \Rightarrow 1 = c_1 e^0 + c_2 e^0 \Rightarrow c_1 + c_2 = 1$

$y'(x) = \pi c_1 e^{\pi x} - \pi c_2 e^{-\pi x}$

$y'(0) = -\pi \Rightarrow -\pi = \pi c_1 e^0 - \pi c_2 e^0 \Rightarrow c_1 - c_2 = -1$

so $c_1 = 0, \; c_2 = 1$

$\therefore$ the unique solution is $\boxed{y(x) = e^{-\pi x}}$

(b) $y'' - 4y' + 20y = 0, \quad y(0) = 1, \quad y'(0) = 18$

char. eq. is $\lambda^2 - 4\lambda + 20 = 0$

the roots are $\lambda_{1,2} = \dfrac{4 \pm \sqrt{(-4)^2 - 4(20)}}{2} = \dfrac{4 \pm \sqrt{-64}}{2} = 2 \pm 4i$

so the general solution is $y(x) = c_1 e^{2x} \cos(4x) + c_2 e^{2x} \sin(4x)$

$y(0) = 1 \Rightarrow 1 = c_1 e^0 \cos(0) + c_2 e^0 \sin(0) \Rightarrow c_1 = 1$

$y'(x) = 2c_1 e^{2x} \cos(4x) - 4c_1 e^{2x} \sin(4x) + 2c_2 e^{2x} \sin(4x)$
$+ 4c_2 e^{2x} \cos(4x)$

$y'(0) = 18 \Rightarrow 18 = 2c_1 + 4c_2 \Rightarrow c_2 = 4$

$\therefore$ the unique solution is $\boxed{y(x) = e^{2x} \cos(4x) + 4e^{2x} \sin(4x)}$

(B)

Question 4 (5 marks) Use Newton's Method to find the point of intersection of $p(x) = 9 - x^2$ and $q(x) = \ln x$ to 6 decimal places. Start with $x_0 = 1$.

let $f(x) = 9 - x^2 - \ln x$

then $f'(x) = -2x - \frac{1}{x}$

so $x_{n+1} = x_n - \dfrac{9 - x_n^2 - \ln x_n}{-2x_n - \frac{1}{x_n}}$

$= x_n + \dfrac{9 - x_n^2 - \ln x_n}{2x_n + \frac{1}{x_n}}$

$= \dfrac{x_n^2 + 10 - \ln x_n}{2x_n + \frac{1}{x_n}}$

$x_0 = 1, \quad x_1 = \dfrac{1 + 10 - \ln(1)}{2 + \frac{1}{1}} = \dfrac{11}{3} = 3.666667$

$x_2 = 2.911515$

$x_3 = 2.823038$

$x_4 = 2.821812 = x_5 \quad \therefore$ stop

$p(2.821812) = 1.037377$
$q(2.821812) = 1.037379$) okay

$\boxed{(2.821812,\ 1.037378)}$