

MAT 2322 Assignment #1: Solutions

1. $f(x,y) = 6x^2 - 2x^3 + 3y^2 + 6xy$
so $f_x = 12x - 6x^2 + 6y = 6(2x - x^2 + y)$
and $f_y = 6y + 6x = 6(y+x)$
then $f_y = 0 \Rightarrow 6(y+x) = 0 \Rightarrow y+x = 0 \Rightarrow y = -x$
and so $f_x = 0 \Rightarrow 6(2x - x^2 + y) = 0 \Rightarrow 2x - x^2 + y = 0$
plug in $y = -x$, we get $2x - x^2 - x = 0$ or $x - x^2 = 0$
so $x(x-1) = 0 \Rightarrow x = 0$ or $x = 1$
and so the critical points are $(0,0)$ and $(1,-1)$
$$\left. \begin{array}{l} f_{xx} = 12 - 12x \\ f_{yy} = 6 \\ f_{xy} = 6 \end{array} \right\} \text{ so } D = f_{xx}f_{yy} - (f_{xy})^2 = (12-12x)(6) - (6)^2$$
$$= 36 - 72x = 36(1-2x)$$

then $D(0,0) = 36 > 0$, $f_{xx}(0,0) = 12 > 0 \Rightarrow (0,0)$ is a local min
and $D(1,-1) = -36 < 0 \Rightarrow (1,-1)$ is a saddle point

2. $f(x,y) = e^{x^2+y^2-4x}$
so $f_x = (2x-4)e^{x^2+y^2-4x}$
and $f_y = 2ye^{x^2+y^2-4x}$
 $f_x = 0 \Rightarrow 2x-4=0 \Rightarrow x=2$
 $f_y = 0 \Rightarrow 2y=0 \Rightarrow y=0$ } so $(2,0)$ is the only critical point
 $f_{xx} = 2e^{x^2+y^2-4x} + (2x-4)^2 e^{x^2+y^2-4x}$
$$= (4x^2 - 16x + 18)e^{x^2+y^2-4x}$$
 $f_{yy} = 2e^{x^2+y^2-4x} + (2y)^2 e^{x^2+y^2-4x} = (4y^2 + 2)e^{x^2+y^2-4x}$ $f_{xy} = 2y(2x-4)e^{x^2+y^2-4x}$
$$D = [(4y^2+2)(4x^2-16x+18) - 4y^2(2x-4)^2] e^{2(x^2+y^2-4x)}$$
 $D(2,0) > 0$, $f_{xx}(2,0) > 0 \Rightarrow (2,0)$ is a local min

3. $f(x, y) = (4x - x^2) \cos y$ on $1 \leq x \leq 3, -\pi/4 \leq y \leq \pi/4$

look for critical points in the interior:

$$f_x = (4 - 2x) \cos y = 2(2 - x) \cos y$$

$$f_y = -(4x - x^2) \sin y$$

then $f_x = 0 \Rightarrow 2(2 - x) \cos y = 0 \Rightarrow x = 2$ or $\cos y = 0$

but $\cos y > 0$ on $-\pi/4 \leq y \leq \pi/4 \Rightarrow x = 2$ only

and $f_y = 0 \Rightarrow -(4x - x^2) \sin y = 0$

since $x = 2, 4x - x^2 \neq 0$, so must have $\sin y = 0$

and hence $y = 0$

so there is one critical point at $(2, 0)$, which is in the interior of the region, $f(2, 0) = (4(2) - (2)^2) \cos(0) = 4$

along the boundary:

if $x = 1$, then $f(1, y) = 3 \cos y$

if $x = 3$, then $f(3, y) = 3 \cos y$

so we look for max and min of $g(y) = 3 \cos y$ on $[-\pi/4, \pi/4]$

$$g(-\pi/4) = g(\pi/4) = 3/\sqrt{2}$$

$$g'(y) = -3 \sin y = 0 \text{ if } y = 0 \quad g(0) = 3$$

if $y = -\pi/4$, then $f(x, -\pi/4) = (4x - x^2)/\sqrt{2}$

if $y = \pi/4$, then $f(x, \pi/4) = (4x - x^2)/\sqrt{2}$

so we look for max and min of $h(x) = (4x - x^2)/\sqrt{2}$ on $[1, 3]$

$$h(1) = h(3) = 3/\sqrt{2}$$

$$h'(x) = (4 - 2x)/\sqrt{2} = 0 \text{ if } x = 2, \quad h(2) = 4/\sqrt{2} = 2\sqrt{2}$$

\therefore abs max is 4 at $(2, 0)$

abs min is $3/\sqrt{2}$ at $(1, -\pi/4), (1, \pi/4), (3, -\pi/4), (3, \pi/4)$

4. $f(x,y) = x^2 + y^2$ subject to $g(x,y) = x^2 - 2x + y^2 - 4y = 0$

$$\nabla f = \lambda \nabla g \Rightarrow f_x = \lambda g_x \Rightarrow 2x = \lambda(2x-2)$$

$$f_y = \lambda g_y \Rightarrow 2y = \lambda(2y-4)$$

or $x = \lambda(x-1)$ $x=0 \Rightarrow \lambda=0$
 $y = \lambda(y-2)$ $y=0 \Rightarrow \lambda=0$

if $x=1, y=0$ * } $\Rightarrow x \neq 1, y \neq 2$
 if $y=2, x=0$ *

thus $\lambda = \frac{x}{x-1}$ and $\lambda = \frac{y}{y-2}$

$$\text{so } \frac{x}{x-1} = \frac{y}{y-2} \Rightarrow xy - 2x = xy - y \Rightarrow y = 2x$$

plug into $g(x,y) = x^2 - 2x + (2x)^2 - 4(2x) = 0$

$$\text{or } x^2 - 2x + 4x^2 - 8x = 5x^2 - 10x = 5x(x-2) = 0$$

then $x=0 \Rightarrow (0,0)$ or $x=2 \Rightarrow (2,4)$

$$f(0,0) = 0$$

$$f(2,4) = (2)^2 + (4)^2 = 20$$

\therefore the max is 20 at (2,4) and the min 0 at (0,0)

5. the distance of point (x,y,z) to the origin is

$$d(x,y,z) = \sqrt{x^2 + y^2 + z^2}$$

but the point that minimizes this distance will also minimize its square $d^2(x,y,z) = x^2 + y^2 + z^2$ (which is much easier to work with)

so we minimize $x^2 + y^2 + z^2 = f(x,y,z)$ subject to the constraints

$$g_1(x,y,z) = y + 2z = 12 \text{ and } g_2(x,y,z) = x + y = 6$$

then $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$ will give us

$$f_x = \lambda_1 g_{1x} + \lambda_2 g_{2x} \Rightarrow 2x = \lambda_2$$

$$f_y = \lambda_1 g_{1y} + \lambda_2 g_{2y} \Rightarrow 2y = \lambda_1 + \lambda_2$$

$$f_z = \lambda_1 g_{1z} + \lambda_2 g_{2z} \Rightarrow 2z = 2\lambda_1$$

so we have that $\lambda_2 = 2x$ and $\lambda_1 = z$

$$\text{so } 2y = \lambda_1 + \lambda_2 \text{ becomes } 2y = z + 2x \text{ or } 2x - 2y + z = 0$$

then we solve $2x - 2y + z = 0$ ①

$$y + 2z = 12$$
 ②

$$x + y = 6$$
 ③

$$\text{①} - 2 \times \text{③} \Rightarrow -4y + z = -12$$
 ④

$$\text{then } 4 \times \text{②} + \text{④} \Rightarrow 9z = 36 \Rightarrow z = 4 \Rightarrow y = 4 \Rightarrow x = 2$$

\therefore the point is $(2, 4, 4)$

(this point will give the min as there is no max in this situation)