SOME EXERCISES IN PREPARATION OF THE MIDTERM

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Exercise. *Find the numbers a and b so that the function*

$$f(x) = \begin{cases} \frac{1}{1-x} & \text{if } x < 0 \\ ax + b & \text{if } 0 \le x \le 1 \\ x & \text{if } x \ge 1 \end{cases}$$

is continuous.

Solution: The function $f(x)$ is the result of three pieces of function. All three pieces are continuous, therefore there might be continuity problems only between one piece and the other, that is, when $x = 0$ and when $x = 1$. A function f is continuous at $x = a$ if the following conditions are satisfied

(1) f is defined at a,
(2) $\lim_{x\to a} f(x)$ exists[1],
(3) $\lim_{x\to a} f(x) = f(a)$.

Continuity at $x = 0$: The function f is defined at $x = 0$, hence condition (1) is satisfied. We want to verify for which values of a and b condition (2) is satisfied. Let's compute both $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$.

$$\lim_{x\to 0^-} f(x) =^{2} \lim_{x\to 0^-} \frac{1}{1-x} = -1.$$

On the other hand

$$\lim_{x\to 0^+} f(x) =^{3} \lim_{x\to 0^+} ax + b = b.$$

Condition (2) therefore translates as

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x),$$

that is, we obtain the condition

$$-1 = b.$$

Hence we obtain that f is continuous at $x = 0$ if $b = -1$.
Continuity at $x = 1$: The function f is defined at $x = 1$, hence condition (1) is satisfied. We want to verify for which values of a condition (2) is satisfied. Let's

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[1]Recall that $\lim_{x\to a} f(x)$ exists and is equal to L if and only if

$$\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x).$$

[2]The symbol $x \to 0^-$ implies that x is approaching 0 *from the left*, that is, x is very close to 0, yet *smaller*

[3]The symbol $x \to 0^+$ implies that x is approaching 0 *from the right*, that is, x is very close to 0, yet *bigger*

compute both $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.

$$\lim_{x\to 1^-} f(x) =^{4} \lim_{x\to 1^-} ax - 1 = a - 1.$$

On the other hand

$$\lim_{x\to 1^+} f(x) =^{5} \lim_{x\to 1^+} x = 1.$$

Condition (2) therefore translates as

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x),$$

that is, we obtain the condition

$$a - 1 = 1.$$

Hence we obtain that f is continuous at $x = 1$ if $a = 2$.

We conclude therefore that f is continuous if $a = 2$ and $b = -1$.

Exercise. *Consider the function*

$$f(x) = \begin{cases} x + a & \text{if } x \le 1 \\ ax^2 + b & \text{if } x > 1 \end{cases}$$

Find the values of a and b such that $f(x)$ is continuous everywhere.

Exercise. *Find the following limits - if they exist - for the function*

$$f(x) = \frac{1}{(x-1)(x+2)},$$

explain you answeers in a sentence or two:

$$\lim_{x\to -2^-} f(x), \qquad \lim_{x\to -2^+} f(x),$$
$$\lim_{x\to 1^+} f(x), \qquad \lim_{x\to 0} f(x),$$
$$\lim_{x\to +\infty} f(x), \qquad \lim_{x\to -\infty} f(x),$$

and the equations of all asymptotes.

Part of the solution:

$\lim_{x\to -2^-} f(x)$:

$$\lim_{x\to -2^-} \frac{1}{(x-1)(x+2)} = \frac{\lim_{x\to -2^-} 1}{\lim_{x\to -2^-}(x-1)\lim_{x\to -2^-}(x+2)} = \frac{1}{1\cdot(-2^- + 2)} = \frac{1}{0^-} =^{6} -\infty$$

.

$\lim_{x\to -\infty} f(x)$:

$$\lim_{x\to -\infty} \frac{1}{(x-1)(x+2)} = \lim_{x\to -\infty} \frac{\lim_{x\to -\infty} 1}{(\lim_{x\to -\infty} x - 1)(\lim_{x\to -\infty} x + 2)} = \frac{1}{(-\infty - 1)(-\infty + 2)} = \frac{1}{+\infty} = 0.$$

[4] The symbol $x \to 1^-$ implies that x is approaching 1 *from the left*, that is, x is very close to 1, yet *smaller*

[5] The symbol $x \to 1^+$ implies that x is approaching 1 *from the right*, that is, x is very close to 1, yet *bigger*

[6] the symbol 0^- means that it is something very close to 0, yet smaller, hence negative

We have seen in class that using limit laws (section 2.3) is not always enough to evaluate limits. Sometimes by using the limit laws we may obtain results that don't make sense, the so called *indeterminate forms*:

$$\frac{0}{0}, \qquad \frac{\infty}{\infty}, \qquad +\infty-\infty, \qquad 0\cdot\infty.$$

If, by using the limit laws, we obtain an indeterminate form, we have to do more work in order to obtain a result. The only acceptable answers for a limit are either a number L (which could possilby be 0) or $\pm\infty$.

Exercise. *Evaluate the following limits:*

$$\lim_{t\to 3}\frac{3-t}{5-\sqrt{t^2+16}}, \qquad \lim_{x\to\infty}\frac{2x^{-2}+3x^{-4}}{5x^{-1}-x^{-3}}.$$

Solution:

- Let's try first by using the limit laws:

$$\lim_{t\to 3}\frac{3-t}{5-\sqrt{t^2+16}} = \frac{\lim_{t\to 3}(3-t)}{\lim_{t\to 3}5-\sqrt{t^2+16}} = \frac{0}{0}.$$

This is an *undetermined form*, hence we have to try with another approach. The function contains a square root, hence we can use the *completion of squares*[7]*:*

$$\lim_{t\to 3}\frac{3-t}{5-\sqrt{t^2+16}} = \lim_{t\to 3}\frac{3-t}{5-\sqrt{t^2+16}}\frac{5+\sqrt{t^2+16}}{5+\sqrt{t^2+16}} = \lim_{t\to 3}\frac{(3-t)(5+\sqrt{t^2+16}}{25-(t^2+16)} =$$

$$= \lim_{t\to 3}\frac{(3-t)(5+\sqrt{t^2+16})}{9-t^2} = \lim_{t\to 3}\frac{(3-t)(5+\sqrt{t^2+16})}{(3-t)(3+t)} = \lim_{t\to 3}\frac{5+\sqrt{t^2+16}}{3+t} =$$

$$= \frac{\lim_{t\to 3}5+\sqrt{t^2+16}}{\lim_{t\to 3}3+t} = \frac{10}{6} = \frac{5}{3}.$$

- By using the limit laws we obtain

$$\lim_{x\to\infty}\frac{2x^{-2}+3x^{-4}}{5x^{-1}-x^{-3}} = \lim_{x\to\infty}\frac{2\frac{1}{x^2}+3\frac{1}{x^4}}{5\frac{1}{x}-\frac{1}{x^3}} = \frac{\lim_{x\to\infty}2\frac{1}{x^2}+\lim_{x\to\infty}3\frac{1}{x^4}}{\lim_{x\to\infty}5\frac{1}{x}-\lim_{x\to\infty}\frac{1}{x^3}} = \frac{2\frac{1}{\lim_{x\to\infty}x^2}+3\frac{1}{\lim_{x\to\infty}x^4}}{5\frac{1}{\lim_{x\to\infty}x}-\frac{1}{\lim_{x\to\infty}x^3}} = \frac{0}{0},$$

which is an *undetermined form*. We need therfore to use a different approach. In particular we want to "highlight" the "strongest powers" by factoring them:

$$\lim_{x\to\infty}\frac{2\frac{1}{x^2}+3\frac{1}{x^4}}{5\frac{1}{x}-\frac{1}{x^3}} = \lim_{x\to\infty}\frac{\frac{1}{x^4}(2x^2+3)}{\frac{1}{x^3}(5x^2-1)} = \frac{(2x^2+3)}{x(5x^2+3)} = \lim_{x\to\infty}\frac{x^2(2+\frac{3}{x^2})}{x[x^2(5-\frac{1}{x^2})]} = 0.$$

Exercise. *Evaluate the following limits:*

$$\lim_{x\to 2}\frac{x-2}{x^2+x-6}, \qquad \lim_{x\to 1}\frac{\sqrt{x^2+5x-5}-x}{x-1}, \qquad \lim_{x\to\infty}\frac{1+x+2x^3}{3+2x+x^3}.$$

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[7] Completion of squares:

$$(a-b)(a+b) = a^2-b^2.$$

We want to use this with $a=5$ and $b=\sqrt{t^2+16}$

Exercise. *Evaluate the limit*

$$\lim_{x\to 2^-} \frac{|x-2|}{x-2}.$$

Hint: We need to break the absolute value. Recall that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

and by composition of function, for any function $f(x)$, we have

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

Now, we are considering $\lim_{x\to 2^-} \frac{|x-2|}{x-2}$, that is x is very close to 2, yet a bit smaller. What does this tell us about the sign of $(x-2)$?

Exercise. *Evaluate*

$$\lim_{x\to +\infty} (\sqrt{x^2-9x+1} - x), \qquad \lim_{x\to -\infty} (\sqrt{x^2-9x+1} - x).$$

Part of solution: By using the limit laws we see that $\lim_{x\to -\infty} (\sqrt{x^2-9x+1} - x) = +\infty$, which is an acceptable answer. On the other hand, if we use the limit laws with the first limit, we obtain

$$\lim_{x\to +\infty} (\sqrt{x^2-9x+1} - x) = +\infty - \infty,$$

which is an indeterminate form. We need to use a different approach. Hint: use a suitable completion of squares!

Exercise.
(1) *Let $f(x) = \ln(1+x^4)$ and $g(x) = \sqrt{16+x}$. Find $f \circ g$ and $g \circ f$ and determine the domain of each of the two composite functions.*
(2) *Find the range of the function $f(x) = e^{2x} + 3$, the inverse function f^{-1} and the range of f^{-1}.*

Exercise. *Find the derivatives of the following functions*

(1) $f(x) = \frac{1}{5x+2}$,
(2) $f(x) = \sin^2(x)$,
(3) $f(x) = \sin(x^2)$,
(4) $f(x) = \cos(\pi x)$,
(5) $f(x) = \cos(\pi)$.

Hints: these are all composite functions. The first step is therefore to identify them as composite functions. for example:

$$f(x) = \frac{1}{5x+2} = (g \circ h)(x)$$

with

$$h(x) = 5x + 2, \qquad \frac{1}{x}.$$

We then apply the *chain rule*:

$$f'(x) = g'(h(x)) \cdot h'(x).$$

We have

$$f(x) = \sin^2(x) = (g \circ h)(x),$$

with $g(x) = x^2$ and $h(x) = \sin(x)$.

Exercise. *Find the derivatives of the following functions*

- $f(x) = \frac{2\sqrt{x} - 3\sqrt[3]{x^2} + 4\sqrt[4]{x^3}}{x^{1/12}}$,
- $f(x) = e^{\sin(x)} + e^x \sin(x)$,
- $f(x) = \frac{\sin(x^2+x+1)}{x^2+x+1}$.

Exercise. *Write the equation of the tangent line to the curve*

$$y = e^{x-2}\sqrt{x-1}$$

at the point $(2, f(2))$.

Solution: The equation of a line has form

$$y = mx + q,$$

where m is the *slope* of the line and q is its *intersection with the* y*-axis.*
Computation of m*:* We know that the slope of a the tangent line to a curve described by a function $f(x)$ at a point $(a, f(a))$ is given by $f'(a)$, that is, the derivative of f, evaluated at a. Let us therefore compute $f'(x)$:

$$f'(x) = {}^{8}(e^{x-2})'(x-1)^{\frac{1}{2}} + e^{x-2}((x-1)^{\frac{1}{2}})' = {}^{9}e^{x-2}(x-1)^{\frac{1}{2}} + e^{x-2}(\frac{1}{2})(x-1)^{-\frac{1}{2}}.$$

Hence we obtain that

$$f'(2) = e^0(1)^{\frac{1}{2}} + e^0(\frac{1}{2})(1)^{-\frac{1}{2}} = \frac{3}{2}.$$

Hence, the slope of the tangent line to the curve at $(2, f(2))$ is $m = \frac{3}{2}$.
Computation of q*:*[10] We want to compute q, appearing in the equation

$$y = \frac{3}{2}x + q.$$

Remark that we know a point belonging to this line, namely the point $(2, f(2))$. We can therefore put

$$f(2) = \frac{3}{2}(2) + q$$

and solve the equation for q. Now,

$$f(2) = e^0(1)^{\frac{1}{2}} = 1,$$

hence we obtain

$$1 = \frac{3}{2}(2) + q,$$

that is,

$$1 - 3 = q,$$

[8] We are using the *product rule* since $f(x)$ is the product $(e^{x-2}) \cdot (x-1)^{\frac{1}{2}}$.
[9] We use the chain rule several times
[10] A smarter, simpler way that the one showed in class.

and hence that $q = -2$. Hence the equation of the tangent line to the curve at the point $(2, (f(2))$ is

$$y = \frac{3}{2}x - 2.$$

Exercise. *Find the second derivative of the function*

$$f(x) = \frac{\sin(x)}{1 + \cos(x)},$$

and calculate its value at $x = 0$, that is $f''(0)$.

Exercise.
(1) *Suppose $f(x) = \sqrt{x-1}$, $g(x) = x^2 + 2$ and $h(x) = \ln(x+3)$. Find $f \circ g \circ h$ and its domain.*
(2) *Determine the domain and the range of the function $f(x) = \sqrt[3]{2^x + 8}$. Find also $f^{-1}(x)$ and its domain and range.*

Exercise.
(1) *Let $f(x) = x^2 + 1$ and $g(x) = \sin x$. Find $f \circ g$ and $g \circ f$.*
(2) *Find the domain and range of $f(x) = e^{2x} - 1$ and the domain and range of its inverse.*

Solution of (2):

Inverse of $f(x)$: In order to find the inverse of $f(x)$ we need to put

$$y = e^{2x} - 1$$

and solve the equation for x. We have

$$y + 1 = e^{2x},$$

and hence, by taking the *natural logarithm* we obtain

$$\ln(y+1) = 2x,$$

that is

$$x = \frac{\ln(y+1)}{2}.$$

By inverting x and y we obtain

$$f^{-1}(x) = \frac{\ln(x+1)}{2}.$$

Domain and range of $f(x)$: $f(x)$ is an exponential function, hence its domain is $(-\infty, +\infty)$ and its range is $(0, +\infty)$.
Domain and range of $f(x)^{-1}$: We have restrictions on the logarithm, namely the argument of the natural logarithm ln has to be strictly positive. We therefore have the condition

$$y + 1 > 0,$$

that is, the domain of $f(x)^{-1}$ is $(-1, +\infty)$. The range of the logarithm is $(-\infty, +\infty)$.