Concordia University

Course	Number	Sections
ENGR	233	P, Q

Examination	Date	Time	Total Marks	Pages
Final	April 2008	3 hours	100	2

Course Coordinator	Instructors
R. Bhat	M. Bertola, R. Bhat, C. Cummins, G. Dafni, C. David, S. Li

Special Instructions: use of calculators and outside materials is NOT permitted.

Each problem is worth 10 marks unless stated otherwise.

Problem 1. For the vector field

$$\vec{F}(x,y,z) = x^2y\mathbf{i} + xy^2\mathbf{j} + 2xyz\mathbf{k}$$

compute –**if possible**– the following quantities. If it is not possible **explain why not**.

(**a**) $\operatorname{div}(\operatorname{curl}\vec{F}(x,y,z))$, (**b**) $\operatorname{curl}(\operatorname{div}\vec{F}(x,y,z))$, (**c**) $\operatorname{grad}(\operatorname{div}\vec{F}(x,y,z))$, (**d**) $\operatorname{div}(\operatorname{grad}\vec{F}(x,y,z))$

Problem 2. Find the equation of the tangent plane of the surface defined by

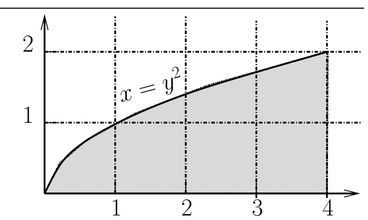
$$z^3 - xyz = 1$$

at the point $(4, \frac{1}{2}, -1)$.

Problem 3.
Evaluate the following integral by reversing the order of integration

$$\int_0^2 \int_{y^2}^4 e^{\sqrt{x^3}} dx dy$$

[Hint: the following substitution may be of help: $u = x^{\frac{3}{2}}$]



Problem 4. Find the rate of change at the point $(2,1,3)$ of the following function $f(x,y,z) = \frac{xy}{z^2}$ along the directions given by unit vectors parallel to
(a) $\mathbf{i}$; **(b)** $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Problem 5. Using **Stokes' theorem**, compute the flux of the curl of the vector field

$$\vec{F}(x,y,z) = 6yz\mathbf{i} - 24x\mathbf{j} + yze^{x^2+\arctan(z)}\mathbf{k}$$

across the surface S (oriented upwards) of the paraboloid $z = y^2 + x^2$, $z \leq 4$, with boundary the circle $z = 4, x^2 + y^2 = 4$.

Problem 6. Find the mass $M = \iiint_{\mathcal{R}} \rho(x,y,z)dV$ of the solid in the first octant (namely $x \geq 0, y \geq 0,\ z \geq 0$) bounded by the coordinate planes and the graph of $x + y + z = 1$ if the density is given by $\rho = x + 2y$.

Problem 7.
Evaluate the work done by the **conservative** force

$$\vec{F}(x, y) = y\mathrm{e}^{xy}\mathbf{i} + (x\mathrm{e}^{xy} + 2y)\mathbf{j}$$

along any path that joins the starting point $(0, 0)$ and ending point $(1, 2)$. **You must use the potential function**.

Problem 8. Find the curvature $\varkappa(t)$ and the components of the acceleration $a_N(t), a_T(t)$ for the curve described by

$$\mathbf{r}(t) = (t + 1)\mathbf{i} + (t^2 - t)\mathbf{j} + \mathrm{e}^{-t}\mathbf{k}$$

Problem 9. Use **Green's theorem** to compute the line-integral

$$\oint_C y^2 \mathrm{d}x + x\mathrm{d}y$$

where C is the boundary of the region determined by the graphs of $x = 0$, $x^2 + y^2 = 4$ and with $x \geq 0$.

Problem 10. Use the **Divergence Theorem** to evaluate the outward flux $\iint_S \vec{F} \cdot \vec{n}\mathrm{d}S$ of the given vector field across the surface specified

$$\vec{F}(x, y, z) = x^3\mathbf{i} + (y^3 + xz)\mathbf{j} + (z^3 + z^2)\mathbf{k}$$

$$x^2 + y^2 + z^2 = a^2, \quad a > 0.$$

$\cos\theta = \vec{a} \bullet \vec{b} / (\|\mathbf{a}\| \|\mathbf{b}\|)$

$\text{comp}_{\mathbf{b}}\mathbf{a} = \|\mathbf{a}\| \cos\theta = \mathbf{a} \cdot \hat{\mathbf{b}}$

$\text{proj}_{\mathbf{b}}\mathbf{a} = (\vec{\mathbf{a}} \cdot \hat{\mathbf{b}})\hat{\mathbf{b}}$

Area of a parallelogram = $\|\mathbf{a} \times \mathbf{b}\|$

Volume of a parallelepiped = $|\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})|$

Equation of a line:

$\vec{r} = \vec{r}_2 + t(\vec{r}_2 - \vec{r}_1) = \vec{r}_2 + t\vec{a}$

Equation of a plane: $a\,x + b\,y + cz + d = 0$

also: $[(\vec{r}_2 - \vec{r}_1) \times (\vec{r}_3 - \vec{r}_1)] \bullet (\vec{r} - \vec{r}_1) = 0$

$$\frac{d\vec{r}(s)}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt}$$

Length of a curve : $s = \int_{t_1}^{t_2} |\vec{r}\,'(t)|\,dt$

$$\kappa = \left\|\frac{d\vec{\mathbf{T}}}{ds}\right\| = \left\|\frac{d^2\vec{\mathbf{r}}}{ds^2}\right\| = \frac{|\vec{\mathbf{T}}'|}{|\vec{\mathbf{r}}'|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

$$\vec{\mathbf{a}}(t) = \kappa v^2 \hat{\mathbf{N}} + \frac{dv}{dt}\hat{\mathbf{T}} = a_N \hat{\mathbf{N}} + a_T \hat{\mathbf{T}}$$

$$\hat{\mathbf{N}} = \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}$$

$$\hat{\mathbf{T}} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

The Binormal

$$\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}$$

$$a_T = \frac{dv}{dt} = \frac{\|\mathbf{v} \bullet \mathbf{a}\|}{\|\mathbf{v}\|} \quad \& \quad a_N = kv^2 = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|}$$

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \quad \& \quad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$$

$D_u(F) = \nabla F \bullet \hat{u}$, $\hat{u}$ = unit vector

Equation of Tangent Plane: $\vec{n}_o \bullet (\vec{r} - \vec{r}_o) = 0$, $\vec{n}_o = \nabla F$ at P

$$W = \int_C \vec{F} \bullet d\vec{r}$$

equation of normal line to a surface: $\vec{n}_o \times (\vec{r} - \vec{r}_o) = 0$, $\vec{n}_o = \nabla F$ at P

$$\int_C F(x,y)ds = \int_a^b F(f(t), g(t))\sqrt{[f']^2 + [g']^2}\,dt = \int_a^b F(x, f(x))\sqrt{1 + [f']^2}\,dx$$

$$\oint_C \vec{F} \bullet d\vec{r} = \iint_S (curl\,\vec{F}) \bullet \hat{n}dS$$

$$\oiint_S (\vec{F} \bullet \hat{n})dS = \iiint_D (div\,\vec{F})dV$$

$$\oint_C [Pdx + Qdy] = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]dxdy$$

$$\tilde{x} = \frac{\iiint_D x\rho(x,y,z)dV}{m},$$

$$m = \iiint_D \rho(x,y,z)dV$$

$$I_x = \iiint_D (y^2 + z^2)\rho(x,y,z)dV;$$

$x = r\cos\theta$, $y = r\sin\theta$; $z = z$; $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(y/x)$

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)}$$

$x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$,

$\rho = \sqrt{x^2 + y^2 + z^2}$, $\theta = \tan^{-1}(y/x)$, $\phi = \tan^{-1}(\sqrt{x^2 + y^2}/z)$

$dV = r\;dr\,d\theta\,dz$

$dV = \rho^2\sin\phi\;d\rho d\phi\,d\theta$