Concordia University	Department of Mathematics and Statistics

Course	**Number**	**Sections**
EMAT	233	P, Q

Examination	**Date**	**Time**	**Total Marks**	**Pages**
Final	December 15, 2005	3 hours	100	2

Course Examiner	**Instructors**
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Special Instructions: use of calculators and outside materials is NOT permitted.

Each problem is worth 10 marks.
Write your answers in the examination booklet. Write clearly and neatly and show all your work.
In order to receive full marks, you must justify your answers fully.
NOTE: You may use the answers to some problems in other problems. You may use symmetry and geometrical considerations to simplify your computations, as long as these are justified.

1. (a) Identify the formula for the **curvature** of the curve traced by the motion of a particle with position vector $\vec{r}(t)$, velocity $\vec{v}(t)$, and acceleration $\vec{a}(t)$:

$$\kappa = \frac{\vec{v}(t) \cdot \vec{a}(t)}{\|\vec{v}(t)\|} \quad \text{or} \quad \kappa = \frac{\|\vec{v}(t) \times \vec{a}(t)\|}{\|\vec{v}(t)\|} \quad \text{or} \quad \kappa = \frac{\|\vec{v}(t) \times \vec{a}(t)\|}{\|\vec{v}(t)\|^3}.$$

   (b) Find the curvature of the curve C given by the equation

$$\vec{r}(t) = a \cos t \, \mathbf{j} + a \sin t \, \mathbf{k}, \quad 0 \le t \le 2\pi.$$

   (c) Give a simple description of the curve C in part (a). What is its radius of curvature?

2. (a) Let $f(x, y, z)$ be a scalar function with second-order partial derivatives. Show that

$$\operatorname{div}(\operatorname{grad} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

   This can also be written as $\vec{\nabla} \cdot \vec{\nabla} f = \nabla^2 f$, which is known as the Laplacian of f.

   (b) Any scalar function f for which $\nabla^2 f = 0$ is said to be harmonic. Verify that

$$f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$$

   is harmonic when $(x, y, z) \neq (0, 0, 0)$.

3. Find the equation of the tangent plane to the graph of the equation

$$z = 25 - x^2 - y^2$$

   at the point $(3, -4, 0)$.

4. Compute $\operatorname{curl} \vec{F}$ and evaluate the line integrals $\int_C \vec{F} \cdot d\vec{r}$ for the given vector fields $\vec{F}$ and curves C.

   (a) $\vec{F} = (2x + e^{-y})\mathbf{i} + (4y - xe^{-y})\mathbf{j}$; C is given by $y = x^4$, going from the point $(0,0)$ to the point $(1,1)$ in the xy plane.

   (b) $\vec{F} = xy\mathbf{j} - xz\mathbf{k}$; C is the closed curve in the shape of the triangle made up of the line segments going from $(1,0,0)$ to $(0,1,0)$ in the xy plane, from $(0,1,0)$ to $(0,0,1)$ in the yz plane, and from $(0,0,1)$ back to $(1,0,0)$ in the xz plane.

5. Find the mass of a lamina in the shape of the half-disk bounded by the circle $(x-1)^2 + y^2 = 1$ and the line $y = 0$, lying in the first quadrant of the xy plane (i.e. $x \geq 0$, $y \geq 0$), if the density is given by the function $\rho(x,y) = xy$.

6. (a) State Green's theorem.

   (b) Use Green's theorem to compute the line integral $\oint_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y) = 3x^2y^2\mathbf{i} + xy^2\mathbf{j}$, and C is the rectangle with corners $(-1,0)$,$(1,0)$,$(1,1)$, and $(-1,1)$.

7. Define the surface S to be that part of the plane $x + y + z = 1$ which lies in the first octant, (that is, which satisfies $x \geq 0$, $y \geq 0$ and $z \geq 0$), oriented upwards. Let $\vec{F}$ be the vector field $\vec{F}(x,y,z) = z\mathbf{j} + y\mathbf{k}$. Find the flux $\iint_S \vec{F} \cdot \vec{n}\, dS$ of the vector field $\vec{F}$ through the surface S.

8. (a) State Stokes' theorem.

   (b) Define the surface S to be that part of the plane $x + y + z = 1$ which lies in the first octant, (that is, which satisfies $x \geq 0$, $y \geq 0$ and $z \geq 0$), oriented upwards. Let $\vec{F}$ be the vector field $\vec{F}(x,y,z) = xy\mathbf{j} - xz\mathbf{k}$. Verify Stokes' theorem, for this choice of surface S and vector field $\vec{F}$. (HINT: You may use the results of previous questions.)

9. Consider the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the surface of the sphere $x^2 + y^2 + z^2 = 1$. If the volume is $V = \frac{\pi(2-\sqrt{2})}{3}$, find the centre of mass of this solid, assuming the density is constant. (HINT: Use symmetry to evaluate two of the three coordinates of the centre of mass.)

10. (a) State the divergence theorem.

    (b) Define the closed surface S to be the outward oriented boundary of the solid bounded by the graphs $x^2 + y^2 = 1$, $z = -1$, and $z = 1$, and define the vector field $\vec{F}$ by $\vec{F}(x,y,z) = (xz + y^2)\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$. Compute the outward flux $\iint_S \vec{F} \cdot \vec{n}\, dS$ of the vector field $\vec{F}$ through the surface S, using the divergence theorem.