(A)

MAT 1322 A W2012 Wednesday, Feb. 8th 8:30–9:50 Prof. Desjardins

MIDTERM TEST 1

Max = 20

Student Number: Solutions

•Time: 80 min.

• Only basic scientific calculators are permitted (non-graphing, non-programmable, no integration or differentiation capabilities). Notes or books are not permitted.

• Work all problems in the space provided. Use the backs of the pages for rough work if necessary. Do not use any other paper.

• The problems require complete and clearly presented solutions and carry part marks if there is substantial correct work towards the solution.

• There are six questions worth a total of 20 marks.

Ⓐ

1. (2 points) Consider the integral $\int_0^9 \frac{2}{(x-1)^{4/3}}\,dx$.

(a) Is this an improper integral and if yes, why?

yes, type II, integrand unbounded on interval

or integrand has vertical asymptote at $x=1$

(b) Does it converge or diverge? If it converges, give its value. If it diverges, explain why.

$$\int_0^9 \frac{2}{(x-1)^{4/3}}\,dx = \int_0^1 \frac{2}{(x-1)^{4/3}}\,dx + \int_1^9 \frac{2}{(x-1)^{4/3}}\,dx$$

then
$$\int_0^1 \frac{2}{(x-1)^{4/3}}\,dx = \lim_{t\to 1^-} \int_0^t \frac{2}{(x-1)^{4/3}}\,dx$$

$$= \lim_{t\to 1^-} -6(x-1)^{-1/3}\Big|_0^t$$

$$= \lim_{t\to 1^-} -6\left((t-1)^{-1/3} - (-1)\right)$$

$$= \infty \quad \text{(limit does not exist)}$$

So integral **diverges**

Ⓐ

2. (3 points) Determine if the statements below are true or false. Write T or F in the box provided.

(a) To determine if $\int_0^1 \frac{\sqrt{3+x}}{4x+x^2}\,dx$ is convergent or divergent using a comparison, a student proceeded as follows. Are they right?

(i) $0 \leq \frac{\sqrt{3+x}}{4x+x^2} \leq \frac{\sqrt{4}}{4x+x^2} = \frac{2}{4x+x^2} \leq \frac{2}{x^2}$ for all $0 < x \leq 1$. True or False? **T**

(ii) $\int_0^1 \frac{2}{x^2}\,dx = 2\int_0^1 \frac{1}{x^2}\,dx = \infty$. True or False? **T**

(iii) Supposing that statements (i) and (ii) are correct, the student concludes that $\int_0^1 \frac{\sqrt{3+x}}{4x+x^2}\,dx$ is divergent by the comparison test. True or False? **F**

(b) To determine if $\int_1^\infty \frac{\sqrt{3+x}}{4x+x^2}\,dx$ is convergent or divergent using a comparison, a student proceeded as follows. Are they right?

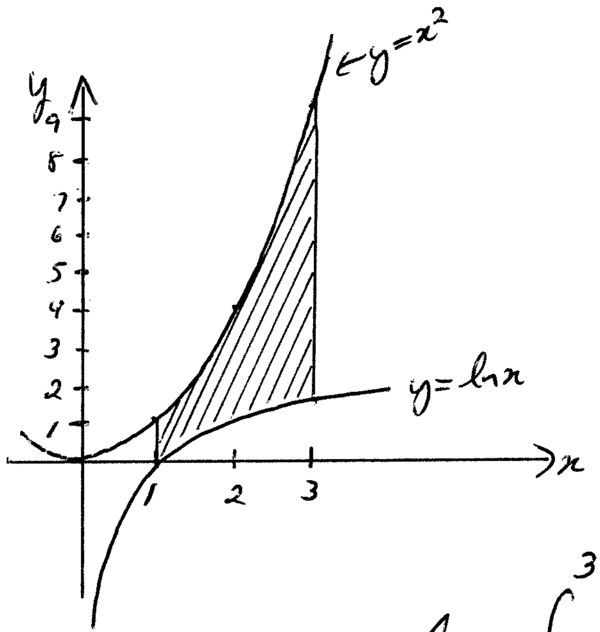
(i) $0 \leq \frac{\sqrt{3+x}}{4x+x^2} \leq \frac{\sqrt{4x}}{4x+x^2} \leq \frac{\sqrt{4x}}{5x^2} = \frac{2}{5x^{3/2}}$ for all $x \geq 1$. True or False? **F**

(ii) $\int_1^\infty \frac{2}{5x^{3/2}}\,dx = \frac{2}{5}\int_1^\infty \frac{1}{x^{3/2}}\,dx = \frac{4}{5}$ True or False? **T**

(iii) Supposing that statements (i) and (ii) are correct, the student concludes that $\int_1^\infty \frac{\sqrt{3+x}}{4x+x^2}\,dx$ is convergent by the comparison test. True or False? **T**

Ⓐ

3. (3 points) Sketch the region bounded by the curves $y = x^2$, $y = \ln x$, $x = 1$ and $x = 3$. What is the area of the region?



area $A = \int_1^3 (x^2 - \ln x)\,dx$

$= \frac{1}{3}x^3 - x\ln x + x \Big|_1^3$

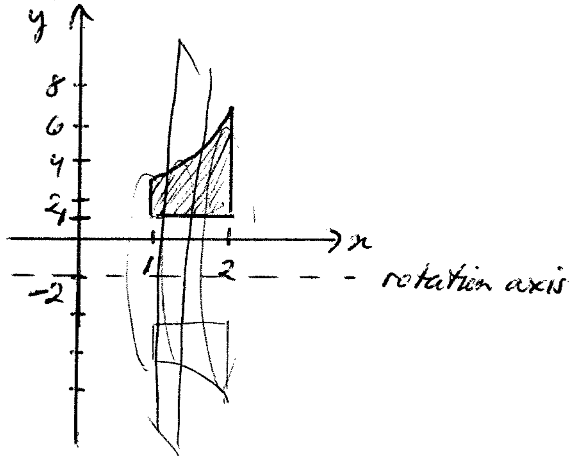
$= (9 - 3\ln 3 + 3) - (\frac{1}{3} - \ln(1) + 1)$

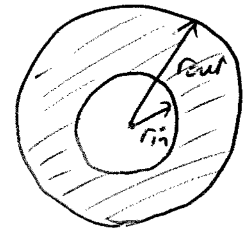
$= \boxed{\frac{32}{3} - 3\ln 3}$

$\simeq \boxed{7.37}$

4. (4 points) Find the volume of the solid obtained when the region bounded by $y = e^x$, $y = 1$, $x = 1$ and $x = 2$ is rotated around the line $y = -2$.

(a) Sketch the region and a typical annular cross-section in a plane perpendicular to the x-axis.





(b) What are the inner radius r_{in}, the outer radius r_{out} and the area $A(x)$ of this ring?

Answers:

$r_{in} = 1 - (-2) = \boxed{3}$

$r_{out} = e^x - (-2) = \boxed{e^x + 2}$

$A(x) = \pi (r_{out}^2 - r_{in}^2) = \pi ((e^x+2)^2 - 3^2) = \boxed{\pi (e^{2x} + 4e^x - 5)}$

(c) Write the integral that gives the volume of the solid and evaluate it.

$$V = \int_1^2 A(x)\,dx = \int_1^2 \pi (e^{2x} + 4e^x - 5)\,dx$$

$$= \pi \left(\frac{1}{2} e^{2x} + 4e^x - 5x \Big|_1^2 \right)$$

$$= \pi \left[\left(\frac{1}{2} e^4 + 4e^2 - 10 \right) - \left(\frac{1}{2} e^2 + 4e - 5 \right) \right]$$

$$= \boxed{\pi \left[\frac{1}{2} e^4 + \frac{7}{2} e^2 - 4e - 5 \right]}$$

$$\simeq \boxed{117.14}$$

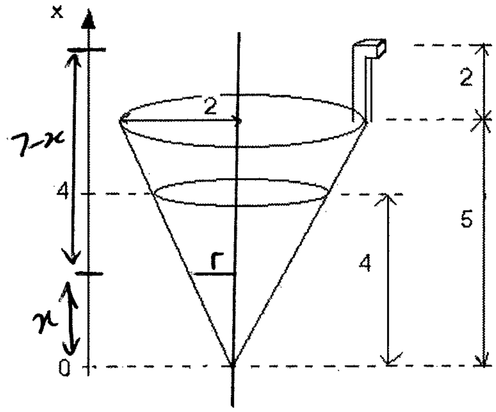
(A)

5. (4 points) A reservoir has the form of a cone as in the diagram.

Its height is 5 m, the radius at the top is 2 m, and it's full of water to a height of 4 m.

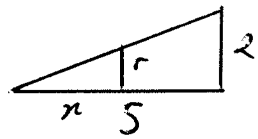
We want to pump all the water to a point 2 m above the reservoir.

Let x be the height measured in metres from the bottom of the reservoir.



(a) What is the radius r of the cone at height x?

Answer: $r =$ $\boxed{\frac{2}{5}x}$



$$\frac{x}{5} = \frac{r}{2}$$

(b) Determine the approximate volume (in m^3) of a thin slice of the water between the heights x and $x + \Delta x$, for small Δx. Give your answer in the form $A(x)\Delta x$.

Answer: Volume $=$ $\pi r^2 \Delta x = \pi\left(\frac{2}{5}x\right)^2 \Delta x = \boxed{\frac{4}{25}\pi x^2 \Delta x}$

(c) Determine the approximate work (in J) required to pump this slice 2 m above the reservoir. Give your answer in the form $P(x)\Delta x$. Recall that the density of water is $\rho = 1000$ kg/m^3 and that the acceleration of gravity is $g = 9.8$ m/s^2.

Answer: Work $=$ $\rho V g\,(7-x) = \boxed{\frac{4}{25}\rho g\,\pi x^2(7-x)\Delta x}$

$$= \boxed{1568\,\pi\, x^2(7-x)\Delta x}$$

(d) Determine the work (in J) required to pump all the water in the reservoir to 2 m above it. Write the appropriate integral and evaluate it.

$$\boxed{W = \int_0^4 1568\,\pi\, x^2(7-x)\,dx}$$

$$= 1568\pi \int_0^4 (7x^2 - x^3)\,dx$$

$$= 1568\pi \left(\frac{7}{3}x^3 - \frac{1}{4}x^4 \Big|_0^4\right)$$

$$= 1568\pi\left(\left(\frac{7}{3}(64) - 64\right) - 0\right)$$

$$= 1568\pi\left(\frac{4}{3}(64)\right)$$

$$\simeq \boxed{133\,803\,\pi \text{ J}}$$

$$\simeq \boxed{420\,353 \text{ J}}$$

Ⓐ

6. (4 points)

(a) Give the integral for the average value of the function $f(x) = x\sin(x^2)$ on the interval $[0, \sqrt{\pi/2}]$ and then evaluate it.

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \boxed{\frac{1}{\sqrt{\pi/2} - 0}\int_0^{\sqrt{\pi/2}} x\sin(x^2)\,dx}$$

$$= \frac{1}{\sqrt{\pi/2}}\left(\frac{-1}{2}\cos(x^2)\Big|_0^{\sqrt{\pi/2}}\right)$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{-1}{2}\right)(0-1)$$

$$= \boxed{\frac{1}{\sqrt{2\pi}}} \simeq \boxed{0.3989}$$

(b) Solve the initial value problem: $\dfrac{dy}{dx} = (1+y^2)\sin x\,, \quad y(0) = 0\,.$

Separate the variables $\quad \dfrac{dy}{1+y^2} = \sin x\,dx$

integrate on both sides $\quad \displaystyle\int \frac{dy}{1+y^2} = \int \sin x\,dx + C$

we get $\quad \arctan y = -\cos x + C$

so $\quad y(x) = \tan(C - \cos x) \quad$ (general solution)

$y(0) = 0 \quad \Rightarrow \quad 0 = \tan(C-1) \quad \Rightarrow \quad C = 1$

$\therefore$ the unique solution is $\boxed{y(x) = \tan(1-\cos x)}$

(B)

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• Work all problems in the space provided. Use the backs of the pages for rough work if necessary. Do not use any other paper.

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• There are six questions worth a total of 20 marks.

Ⓑ

1. (2 points) Consider the integral $\displaystyle\int_0^9 \frac{3}{(x-1)^{5/3}}\,dx$.

(a) Is this an improper integral and if yes, why?

yes, type II, integrand is unbounded on interval

or integrand has vertical asymptote at $x=1$

(b) Does it converge or diverge? If it converges, give its value. If it diverges, explain why.

$$\int_0^9 \frac{3}{(x-1)^{5/3}}\,dx = \int_0^1 \frac{3}{(x-1)^{5/3}}\,dx + \int_1^9 \frac{3}{(x-1)^{5/3}}\,dx$$

So $$\int_0^1 \frac{3}{(x-1)^{5/3}}\,dx = \lim_{t\to 1^-} \int_0^t \frac{3}{(x-1)^{5/3}}\,dx$$

$$= \lim_{t\to 1^-} -\frac{9}{2}(x-1)^{-2/3}\Big|_0^t$$

$$= \lim_{t\to 1^-} -\frac{9}{2}\left[(t-1)^{-2/3} - (1)\right]$$

$$= -\infty \quad \text{(limit does not exist)}$$

$\therefore$ the integral **diverges**

(B)

2. (3 points) Determine if the statements below are true or false. Write T or F in the box provided.

(a) To determine if $\int_0^1 \frac{\sqrt{8+x}}{x+3x^2}\,dx$ is convergent or divergent using a comparison, a student proceeded as follows. Are they right?

(i) $\frac{\sqrt{8+x}}{x+3x^2} \geq \frac{\sqrt{8}}{x+3x^2} \geq \frac{\sqrt{8}}{x}$ for all $0 < x \leq 1$. True or False? F

(ii) $\int_0^1 \frac{\sqrt{8}}{x}\,dx = \sqrt{8}\int_0^1 \frac{1}{x}\,dx = \infty$. True or False? T

(iii) Supposing that statements (i) and (ii) are correct, the student concludes that $\int_0^1 \frac{\sqrt{8+x}}{x+3x^2}\,dx$ is divergent by the comparison test. True or False? T

(b) To determine if $\int_1^\infty \frac{\sqrt{8+x}}{x+3x^2}\,dx$ is convergent or divergent using a comparison, a student proceeded as follows. Are they right?

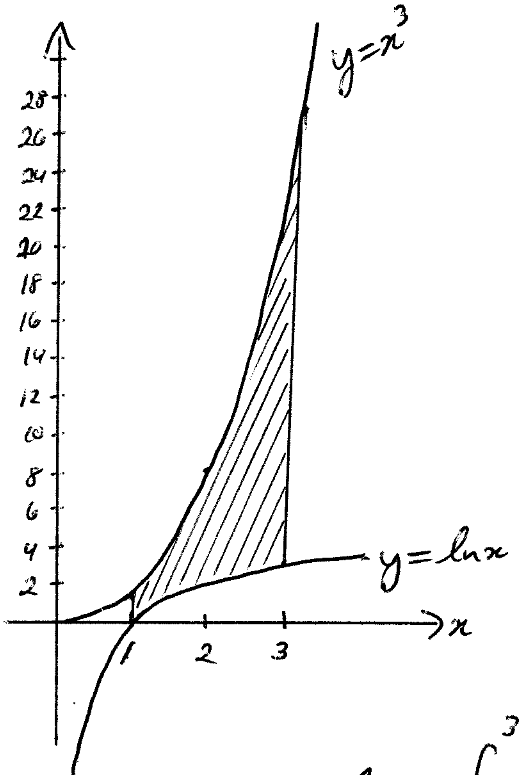
(i) $\frac{\sqrt{8+x}}{x+3x^2} \leq \frac{\sqrt{x}}{x+3x^2} \leq \frac{\sqrt{x}}{3x^2} = \frac{1}{3x^{3/2}}$ for all $x \geq 1$. True or False? F

(ii) $\int_1^\infty \frac{1}{3x^{3/2}}\,dx = \frac{1}{3}\int_1^\infty \frac{1}{x^{3/2}}\,dx = \frac{3}{4}$ True or False? F

(iii) Supposing that statements (i) and (ii) are correct, the student concludes that $\int_1^\infty \frac{\sqrt{8+x}}{x+3x^2}\,dx$ is convergent by the comparison test. True or False? T

(B)

3. (3 points) Sketch the region bounded by the curves $y = x^3$, $y = \ln x$, $x = 1$ and $x = 3$. What is the area of the region?



area $A = \int_1^3 (x^3 - \ln x)\,dx$

$= \frac{1}{4}x^4 - x\ln x + x \Big|_1^3$

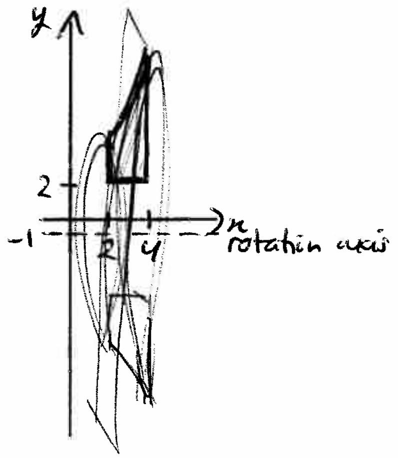
$= \left(\frac{1}{4}(81) - 3\ln 3 + 3\right) - \left(\frac{1}{4} - \ln(1) + 1\right)$

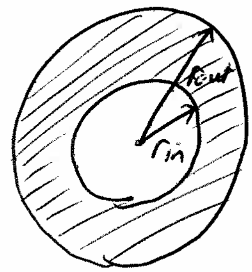
$= \boxed{22 - 3\ln 3}$

$\simeq \boxed{18.7}$

4. (4 points) Find the volume of the solid obtained when the region bounded by $y = e^x$, $y = 2$, $x = 2$ and $x = 4$ is rotated around the line $y = -1$.

(a) Sketch the region and a typical annular cross-section in a plane perpendicular to the x-axis.





(b) What are the inner radius r_{in}, the outer radius r_{out} and the area $A(x)$ of this ring?

Answers:

$r_{in} =$ $2 - (-1) = \boxed{3}$

$r_{out} =$ $e^x - (-1) = \boxed{e^x + 1}$

$A(x) =$ $\pi (r_{out}^2 - r_{in}^2) = \pi ((e^x+1)^2 - 3^2) = \boxed{\pi (e^{2x} + 2e^x - 8)}$

(c) Write the integral that gives the volume of the solid and evaluate it.

Volume $V = \int_2^4 A(x)\,dx$

$= \int_2^4 \pi (e^{2x} + 2e^x - 8)\,dx$

$= \pi \left(\frac{1}{2} e^{2x} + 2e^x - 8x \Big|_2^4 \right)$

$= \pi \left[\left(\frac{1}{2} e^8 + 2e^4 - 32 \right) - \left(\frac{1}{2} e^4 + 2e^2 - 16 \right) \right]$

$= \boxed{\pi \left(\frac{1}{2} e^8 + \frac{3}{2} e^4 - 2e^2 - 16 \right)}$

$\simeq \boxed{4843.1}$

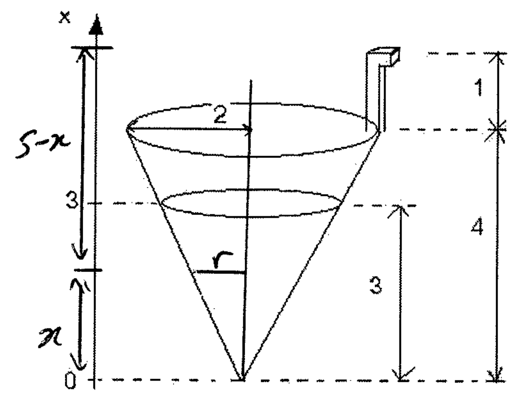
(B)

5. (4 points) A reservoir has the form of a cone as in the diagram.

Its height is 4 m, the radius at the top is 2 m, and it's full of water to a height of 3 m.

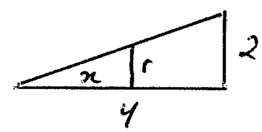
We want to pump all the water to a point 1 m above the reservoir.

Let x be the height measured in metres from the bottom of the reservoir.



(a) What is the radius r of the cone at height x?

Answer: $r = \boxed{\frac{1}{2}x}$



$$\frac{r}{2} = \frac{x}{4}$$

(b) Determine the approximate volume (in m^3) of a thin slice of the water between the heights x and $x + \Delta x$, for small Δx. Give your answer in the form $A(x)\Delta x$.

Answer: Volume = $\pi r^2 \Delta x = \boxed{\frac{1}{4}\pi x^2 \Delta x}$

(c) Determine the approximate work (in J) required to pump this slice 1 m above the reservoir. Give your answer in the form $P(x)\Delta x$. Recall that the density of water is $\rho = 1000$ kg/m^3 and that the acceleration of gravity is $g = 9.8$ m/s^2.

Answer: Work = $\rho g V (5-x) = \boxed{\frac{1}{4}\pi\rho g\, x^2(5-x)\,\Delta x}$

$$= \boxed{2450\,\pi\, x^2(5-x)\,\Delta x}$$

(d) Determine the work (in J) required to pump all the water in the reservoir to 1 m above it. Write the appropriate integral and evaluate it.

$$\boxed{W = \int_0^3 2450\,\pi\, x^2(5-x)\,dx}$$

$$= 2450\,\pi \int_0^3 (5x^2 - x^3)\,dx$$

$$= 2450\,\pi \left(\frac{5}{3}x^3 - \frac{1}{4}x^4 \Big|_0^3\right)$$

$$= 2450\pi \left[45 - \frac{81}{4} - 0\right]$$

$$= 2450\,\pi \left(\frac{99}{4}\right)$$

$$= \boxed{60\,637.5\,\pi \text{ J}}$$

$$\simeq \boxed{190\,498 \text{ J}}$$

6. (4 points)

(a) Give the integral for the average value of the function $f(x) = x\cos(x^2)$ on the interval $[0, \sqrt{\pi/2}]$ and then evaluate it.

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \boxed{\frac{1}{\sqrt{\pi/2}-0}\int_0^{\sqrt{\pi/2}} x\cos(x^2)\,dx}$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{1}{2}\sin(x^2)\Big|_0^{\sqrt{\pi/2}}\right)$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{1}{2}\right)(1-0)$$

$$= \boxed{\frac{1}{\sqrt{2\pi}}} \simeq \boxed{0.3989}$$

(b) Solve the initial value problem: $\dfrac{dy}{dx} = \sqrt{1-y^2}\sin x\,, \quad y(0) = 0\,.$

separate the variables $\dfrac{dy}{\sqrt{1-y^2}} = \sin x\,dx$

integrate on both sides $\displaystyle\int \frac{dy}{\sqrt{1-y^2}} = \int \sin x\,dx + C$

we get $\arcsin y = -\cos x + C$

so $y = \sin(C - \cos x)$ (general solution)

then $y(0) = 0 \Rightarrow 0 = \sin(C-1) \Rightarrow C = 1$

and the unique solution is $\boxed{y(x) = \sin(1-\cos x)}$