CONCORDIA UNIVERSITY

DEPARTMENT OF COMPUTER SCIENCE & SOFTWARE ENGINEERING

COMP 232/4 Mathematics for Computer Science Winter 2016

Assignment 4 Due date: April 15, 2016

1. Write out the addition and multiplication tables for $\mathbb{Z}_6$ (where by addition and multiplication we mean $+_6$ and $\cdot_6$).

Solution:

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$\cdot_6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

2. Let $f : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$ be given by $f(m,n) = (m-n,n)$. The composite functions f_k, for $k \in \mathbb{Z}^+$, are defined as $f_1(m,n) = f(m,n)$, and $f_{k+1}(m,n) = f(f_k(m,n))$, for $k \in \mathbb{Z}^+$. Give a formal proof by induction that $f_k(m,n) = (m-kn,n)$, for all $k \in \mathbb{Z}^+$.

Solution

- Base case: $n = 1$ $f^1(m,n) =_{\mathsf{def.}} f(m,n) =_{\mathsf{def.}} (m-n,n) = (m-1n,n)$.
- Inductive Hypothesis: $n = k$ Assume $f^k(m,n) = (m-kn,n)$.
- Inductive Step: $n = k+1$

$$\begin{aligned} f^{k+1}(m,n) &=_{\mathsf{def.}} \\ f(f^k(m,n)) &=_{\mathsf{ind.hyp.}} \\ f(m-kn,n) &=_{\mathsf{def.}} \\ (m-kn-n,n) &= \\ (m-(k+1)n,n). \end{aligned}$$

3. Use induction to show that for all positive integers n

   (a) $1^3 + 2^3 + 3^3 + \ldots + n^3 = (n(n+1)/2)^2$.

   Solution:

   - Base case: $n = 1$

     LHS = 1, RHS = $(1(1+1)/2)^2 = 1$.

   - Inductive hypothesis: $n = k$

     $\sum_{i=1}^{k} i^3 = (k(k+1)/2)^2$

   - Inductive step: $n = k+1$

$$
\begin{aligned}
\sum_{i=1}^{k+1} i^3 &= \\
\sum_{i=1}^{k} i^3 + (k+1)^3 &= \quad \text{by the inductive hypothesis} \\
(k(k+1)/2)^2 + (k+1)^3 &= \\
(k(k+1)/2)^2 + (k+1)^2(k+1) &= \\
(k+1)^2(k^2/4) + (k+1)^2(k+1) &= \\
(k+1)^2(k^2/4 + k + 1) &= \\
(k+1)^2\left(\frac{k^2+4k+4}{4}\right) &= \\
(k+1)^2\left(\frac{k+2}{2}\right)^2 &= \\
((k+1)(k+2)/2)^2 &
\end{aligned}
$$

(b) $1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n+1)! - 1$

Solution:

- Base case: $n = 1$

  LHS $= 1 \cdot 1! = 1$, RHS $= (1+1)! - 1 = 1$.

- Inductive hypothesis: $n = k$

  $\sum_{i=1}^{k} i \cdot i! = (k+1)! - 1$

- Inductive step: $n = k + 1$

$$\begin{aligned}
\sum_{i=1}^{k+1} i \cdot i! &= \\
\sum_{i=1}^{k} i \cdot i! + (k+1) \cdot (k+1)! &= \quad \text{by the inductive hypothesis} \\
(k+1)! - 1 + (k+1) \cdot (k+1)! &= \\
(k+1)! + (k+1) \cdot (k+1)! - 1 &= \\
(k+1)! \cdot (1 + (k+1)) - 1 &= \\
(k+1)! \cdot (k+2) - 1 &= \\
(k+2) \cdot (k+1)! - 1 &= \\
(k+2)! - 1 &
\end{aligned}$$

(c) if $n > 6$, then $3^n < n!$

Solution:

- Base case: $n = 7$

  LHS $3^7 = 2187 < 5040 = 7! =$ RHS

- Inductive hypothesis: $n = k$

  $3^k < k!$

- Inductive step: $n = k + 1$

  $3^{k+1} \; = \; 3 \cdot 3^k \; < \; 3 \cdot k! \; < \; (k+1) \cdot k! \; = \; (k+1)!$

$$\begin{array}{rcl} 3^{k+1} & = & \text{by arithmetic} \\ 3 \cdot 3^k & < & \text{by the inductive hypothesis} \\ 3 \cdot k! & < & \text{since } k > 6 \text{ we have } 3 < (k+1) \\ (k+1) \cdot k! & = & \text{by arithmetic} \\ (k+1)! & & \end{array}$$

4. Let n be an integer greater than or equal to 2. Prove that a set with n elements has $n(n-1)/2$ subsets containing exactly two elements.

Solution:

Let B be a set and $a \notin B$. We observe that all the 2-subsets of $B \cup \{a\}$ equals all the 2-subsets of B union $\{\{b, a\} : b \in B\}$. In other words

$$\Big\{S : S \in \mathcal{P}(B \cup \{a\}) \text{ and } |S| = 2\Big\} = \Big\{S : S \in \mathcal{P}(B) \text{ and} |S| = 2\Big\} \cup \Big\{\{a, b\} : b \in B\Big\} \qquad (\spadesuit)$$

We can now prove the claim by an induction on $|A|$, that

$$|\{S : S \in \mathcal{P}(B) \text{ and } |S| = 2\}| = n(n-1)/2$$

- Base case: $n = 2$

  Let $A = \{a_1, a_2\}$. Then $\mathcal{P}(A) = \{\emptyset, \{a_1\}, \{a_2\}, \{a_1, a_2 2\}\}$, and there is $2(2-1)/2 = 1$ subset of size 2 in $\mathcal{P}(A)$.

- Inductive hypothesis: $n = k$.

  $A = \{a_1, a_2, \ldots, a_k\}$ and $|\{S : S \in \mathcal{P}(A) \text{ and } |S| = 2\}| = k(k-1)/2$.

- Inductive step: $n = k + 1$

  Let $A = \{a_1, a_2, \ldots, a_k, a_{k+1}\}$, and denote $\{a_1, a_2, \ldots, a_k\}$ by B. Then

$$\begin{aligned} |\{S : S \in \mathcal{P}(A) \text{ and } |S| = 2\}| &= \\ |\{S : S \in \mathcal{P}(B \cup \{a_{k+1}\}) \text{ and } |S| = 2\}| &= \quad \text{by } (\spadesuit) \\ |\{S : S \in \mathcal{P}(B) \text{ and } |S| = 2\}| + |\{\{a_{k+1}, b\} : b \in B\}| &= \\ k(k-1)/2 + k &= \\ (k(k-1) + 2k)/2 &= \\ k(k-1+2)/2 &= \\ k(k+1)/2 & \end{aligned}$$

5. The Fibonacci numbers are defined as: $f_1 = 1$, $f_2 = 1$, and

$$f_n \ = \ f_{n-1} \ + \ f_{n-2} \ , \qquad \text{for} \ \ n \ \geq \ 3 \ .$$

Give a proof by induction to show that $3 \mid f_{4n}$, for all $n \geq 1$.

Solution

- Base case: $n = 1$ $\ f_{4n} = f_4 = f_3 + f_2 = (f_2 + f_1) + f_2 = (1 + 1) + 1 = 3$, and $3|3$.
- Inductive Hypothesis: $n = k$ Assume $3|f_{4k}$.
- Inductive Step: $n = k + 1$

$$\begin{aligned} f_{4(k+1)} &= \\ f_{4k+4} &= \\ f_{4k+3} + f_{4k+2} &= \\ (f_{4k+2} + f_{4k+1}) + f_{4k+2} &= \\ 2 \cdot f_{4k+2} + f_{4k+1}) &= \\ 2 \cdot (f_{4k+1} + f_{4k}) + f_{4k+1}) &= \\ 3 \cdot f_{4k+1} + 2 \cdot f_{4k} & \end{aligned}$$

Obviously $3|\big(3 \cdot f_{4k+1}\big)$. By the inductive hypothesis we have $3|f_{4k}$, so $3|\big(2 \cdot f_{4k}\big)$, and consequently $3|\Big(3 \cdot f_{4k+1} + f_{4k}\Big)$, meaning that $3|f_{4k+1}$.

6. Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. Use strong mathematical induction to show that $P(n)$ is true for $n \geq 18$.

Solution:

- Base cases:
  $P(18)$ is true, since 18 = 7+7+4.
  $P(19)$ is true, since 19 = 7+4+4+4.
  $P(20)$ is true, since 20 = 4+4+4+4+4.
  $P(21)$ is true, since 21 = 7+7+7.

- Strong inductive hypothesis: Let $k \geq 21$ and suppose $P(21), \ldots, P(k)$ all are true.

- Inductive step: We show that $P(k + 1)$ is true.

  By the strong inductive hypothesis $P(k-3)$ is true. Then a $k+1$ cents postage can be formed from the stamps used (by the inductive hypothesis) for $k - 3$ cents by adding a 4 cent stamp. Note that since $k \geq 21$, we have $k - 3 \geq 18$.

7. Give a recursive definition of each of these sets of ordered pairs of positive integers.

   (a) $S = \{(a, b) \ : \ a \in \mathbb{Z}^+, b \in \mathbb{Z}^+, \text{ and } a + b \text{ is odd}\}$

   Solution:

   - Base case 1: $(1, 2) \in S$.
   - Base case 2: $(2, 1) \in S$.
   - Recursive case 1: if $(i, j) \in S$ then $(i + 1, j + 1) \in S$.
   - Recursive case 2: if $(i, j) \in S$ then $(i, j + 2) \in S$.
   - Recursive case 3: if $(i, j) \in S$ then $(i + 2, j) \in S$.

   (b) $S = \{(a, b) \ : \ a \in \mathbb{Z}^+, b \in \mathbb{Z}^+, \text{ and } a | b\}$

   Solution:

   - Base case: $(n, n) \in S$ for all $n \in \mathbb{Z}^+$.
   - Recursive case: if $(i, j) \in S$ then $(i, j \cdot k) \in S$, for all $k \in \mathbb{Z}^+$.

   (c) $S = \{(a, b) \ : \ a \in \mathbb{Z}^+, b \in \mathbb{Z}^+, \text{ and } 3 | (a + b)\}$

   Solution:

   - Base case 1: $(1, 2) \in S$.
   - Base case 2: $(2, 1) \in S$.
   - Recursive case 1: if $(i, j) \in S$ then $(i + 3, j) \in S$.
   - Recursive case 2: if $(i, j) \in S$ then $(i, j + 3) \in S$.
   - Recursive case 3: if $(i, j) \in S$ then $(i + 1, j + 2) \in S$.
   - Recursive case 4: if $(i, j) \in S$ then $(i + 2, j + 1) \in S$.

8. For each of the following relations on the set of all real numbers, determine whether it is reflexive, symmetric, antisymmetric, transitive. Here xRy if and only if:

   (a) $x + 2y = 0$
   (b) $x = 2y$
   (c) $x - y$ is a rational number
   (d) $xy = 0$
   (e) $xy \geq 0$
   (f) $x = 1$ or $y = 1$
   (g) x is a multiple of y
   (h) $xy = 1$

Solution

		reflexive	symmetric	antisymmetric	transitive
a	$x + 2y = 0$	No	No	Yes	No
b	$x = 2y$	No	No	Yes	No
c	$x - y$ is rational	Yes	Yes	No	Yes
d	$xy = 0$	No	Yes	No	No
e	$xy \geq 0$	Yes	Yes	No	No
f	$x = 1$ or $y = 1$	No	Yes	No	No
g	x is a multiple of y	Yes	No	Yes	Yes
h	$xy = 1$	No	Yes	No	No

9. Determine the matrix which represents the transitive closures of the following relations on the set $\{a, b, c, d, e\}$:

   (a) $\{(b, c), (b, e), (c, e), (d, a), (e, b), (e, c)\}$

   (b) $\{(a, b), (a, c), (a, e), (b, a), (b, c), (c, a), (c, b), (d, a), (e, d)\}$

   For each of the two relations the transitive closures can be determined by drawing the graph of the relation, adding the minimal number of arrows to make the relation transitive, and writing down what the transitive closure is. It can also be done systematically (and programmable) using relation matrices, as done below:

   Solution

   In (a) the relation matrix is

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

   Thus, using Boolean arithmetic,

$$R^2 = R \cdot R = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$R^3 = R \cdot R^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

$$R^4 = R \cdot R^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

$$R^5 = R \cdot R^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

   The matrix representing the transitive closure is

$$R^* = R + R^2 + R^3 + R^4 + R^5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

   Thus we can also represent R^* as

$$R^* = \{(b, b), (b, c), (b, e), (c, b), (c, c), (c, e), (d, a), (e, b), (e, c), (e, e)\} .$$

Similarly in (b) we find that

$$R = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad R^* = R + R^2 + R^3 + R^4 + R^5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Thus every element of $\{a, b, c, d, e\}$ is related to itself and to every other element.

10. List all possible relations on the set $\{0, 1\}$ and determine which of these relations are

(a) reflexive (b) symmetric (c) antisymmetric (d) transitive

Solution

There are 16 different relations on the set $\{0, 1\}$, as is most easily seen by listing all possible relation matrices, as done in the Table below:

	R	reflexive	symmetric	antisymmetric	transitive
1	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	No	Yes	Yes	Yes
2	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	No	Yes	Yes	Yes
3	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	No	No	Yes	Yes
4	$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$	No	No	Yes	Yes
5	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	No	No	Yes	Yes
6	$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$	No	No	Yes	Yes
7	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	No	Yes	No	No
8	$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	No	Yes	No	No
9	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	No	Yes	Yes	Yes
10	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	Yes	Yes	Yes	Yes
11	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$	No	No	Yes	Yes
12	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	Yes	No	Yes	Yes
13	$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$	No	No	Yes	Yes
14	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	Yes	No	Yes	Yes
15	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	No	Yes	No	No
16	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	Yes	Yes	No	Yes

11. Give the equivalence classes of the relation

$$aRb \quad \text{if and only if} \quad a^4 \equiv b^4 (\text{mod } 30) \ ,$$

on the set $\{1, 2, 3, ..., 15\}$.

Solution

We can also define this relation as aRb if and only if $a^4 \text{mod } 30 = b^4 \text{mod } 30$. This is indeed an equivalence relation, as it is clearly reflexive, symmetric, and transitive. Computing the values of $f(n) = n^4 \text{mod } 30$, for $n = 1, 2, \cdots, 15$, we find the values in the Table below:

n	$f(n)$	n	$f(n)$	n	$f(n)$	n	$f(n)$
		4	16	8	16	12	6
1	1	5	25	9	21	13	1
2	16	6	6	10	10	14	16
3	21	7	1	11	1	15	15

Thus the equivalence classes are

$$\{1, 7, 11, 13\}, \quad \{6, 12\}, \quad \{10\}, \quad \{15\}, \quad \{2, 4, 8, 14\}, \quad \{3, 9\}, \quad \{5\} \ .$$

12. Are the relations represented by the following matrices equivalence relations?

$$\text{(a)} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution

Both matrices are symmetric, so they represent symmetric relations. Also both matrices only have 1's on the main diagonal, so they represent reflexive relations. To be an equivalence relation a relation must also be transitive. Drawing the graphs of the two relations it is easily seen that both are transitive. This can also be determined by computing the transitive closures, as represented by the relation matrix $R^* = \sum_{k=1}^{n} R^k$, where $n = 4$ for both relation matrices in this problem. For both matrices we find that $R^* = R$, which means that both are transitive, and hence represent equivalence relations.

13. Which of these collections of subsets are partitions of the set of integers?

    (a) the set of even integers and the set of odd integers

    (b) the set of integers divisible by 3, the set of integers leaving a remainder of 1 when divided by 3, and the set of integers leaving a remainder of 2 when divided by 3

    (c) the set of integers less than -100, the set of integers with absolute value not exceeding 100, and the set of integers greater than 100

    (d) the set of integers not divisible by 3, the set of even integers, and the set of integers that leave a remainder of 3 when divided by 6.

Solution

In each of (a), (b), and (c), the subsets form a partition of $\mathbb{Z}$. The subsets in (d) do not form a partition of $\mathbb{Z}$; for example the number 8 belongs to both the subset of even integers and the subset of integers not divisible by 3.