RYERSON UNIVERSITY
DEPARTMENT OF MATHEMATICS

MIDTERM TEST #2 B

MTH 425

Last Name (Prin[redacted] First Na[redacted] Student Numbe[redacted]

Signature:[redacted]
Section Numbe[redacted]
Date: November 7, 2013
Duration: 1.5 hour

**Professor (circle one)**
S. Homayouni
G. Ord

**Instructions:**

1. This is a closed-book test. **Notes, calculators, cell phones and other aids are not permitted.** Students are allowed a **single** $8\frac{1}{2} \times 11$ formula sheet. Verify that your test has pages 1-9.
2. **Section A** is multiple-choice. Circle the correct response. The correct response gets full marks, an incorrect response or no response gets no marks.
3. **Section B** is full-answer.
   (a) Unless otherwise instructed, **make sure you include all significant steps in your solution, presented in the correct order. Unjustified answers will be given little or no credit. Cross out or erase all rough work not relevant to your solution.** Put a box around your final **answer.**
   (b) Write your solutions in the space provided. If you need more space, use the back of the page. Indicate this fact on the original page, making sure that your solution cannot be confused with any rough work which may be there. Marks (out of 70) are shown in brackets.
4. Do not separate the test sheets.

**For Instructor's use only.**

| Page | Mark |
|---|---|
| M/C | 9 |
| 4 | 3 |
| 5 | 8 |
| 6 | 2 |
| 7 | 0 |
| 8 | 8 |
| 9 | 7 |
| Total | 39 |

MTH 425 Midterm 2

Exam, Form: $\boxed{\text{B}}$

Name [redacted]

Student Number: [redacted]

TA: ______________________

Date: ______________________

## Section 1. Identify Test

1. This is the first question of the multiple choice. You have to get this one right! Mark it on your test right away. The large framed letter next to "Exam Form:" above is:
   (a) $\boxed{\text{A}}$
   (b) $\boxed{\text{B}}$
   (c) $\boxed{\text{C}}$

## Section 2. Multiple Choice [ 3 marks each]

2. Given the ordinary differential equation : $(x^2-9)^2y'' + (x+3)y' + 2y = 0$.Consider the following statements:

   (i) There are regular singular points at $x = 0$ and $x = 3$.

   (ii) There is an irregular singular point at $x = 3$ and a regular singular point at $x = -3$.

   (iii) There is a regular singular point at $x = 3$ and an irregular singular point at $x = -3$.

   (iv) There are irregular singular points at $x = 0$ and $x = 3$.

   (a) Only (i) is true.
   (b) Only (ii) is true.
   (c) Only (iii) is true.
   (d) Only (iv) is true.
   (e) None of the above.

3. The Laplace transform of a function $f(t)$ is $F(s) = \frac{1}{(s-a)(s-b)}$ where $a \neq b$. $f(t)$ is:

   (a) $\frac{1}{a-b}(e^{at} - e^{bt})$
   (b) $\frac{1}{a-b}(e^{at} + e^{bt})$
   (c) $\frac{1}{a-b}(e^{at} + te^{bt})$
   (d) $\frac{1}{b-a}(\cosh at)$
   (e) None of the above.

4. Suppose $f(0) = 0$ and the Laplace transform of $f(t)$ is $F(s) = e^{-as}\sin bs$. The transform of $\frac{df}{dt}$ is :

(a) $\quad -ae^{-as}\sin bs$

(b) $\quad -ae^{-as}\sin bs + be^{-as}\cos bs$

(c) $\quad \cancel{s}\, e^{-as}\sin bs$

(d) $\quad t\ e^{-as}\sin bs$

(e) $\quad e^{-as}\sin bs/s$

$\frac{d\, f(t)}{ds} =$

5. Given that $\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}$ then $\mathcal{L}\{t\ e^{-2t}\sin t\}$ equals:

(a) $\quad \frac{1}{(s^2+1)^2}$

(b) $\quad \frac{2s-2}{(s-2)^2+1}$

(c) $\quad \frac{2(s+2)}{((s+2)^2+1)^2}$

(d) $\quad \frac{2}{(s+2)^2+4}$

(e) $\quad$ None of the above.

$\frac{2s}{(s^2+1)^2} = \frac{2(s+2)}{((s+2)^2+1)^2}$

6. If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ and $G = -\frac{dF}{ds}$, then $\mathcal{L}^{-1}\left\{\frac{G(s)}{s+2}\right\}$ equals

(a) $\quad \int_0^t f(\tau)e^{-2(t-\tau)}\,d\tau$

(b) $\quad \int_0^t (\tau - t)\, f(\tau)e^{-2(t-\tau)}\,d\tau$

(c) $\quad \int_0^t \tau\, f(t-\tau)e^{-2(\tau)}\,d\tau$

(d) $\quad \int_0^t \tau\, f(\tau)e^{-2(t-\tau)}\,d\tau$

(e) $\quad$ None of the above.

$L^{-1}\left[-\frac{d\, L\, f(t)}{d\, s\,(s+2)}\right]$

Section 3. Short Answer

7. [10 marks] **Use the definition of the Laplace transform** to calculate the transform of $f(t) = t\cos t$. (i.e. Do not use tables for the transform of $\cos t$.)

$f(t) = t\cos t$

$L\{f(t)\} = \int_0^\infty e^{-st} t\cos t \, dt$

$u = t\cos t\,dt \quad du = \cos t + t\sin t$

$dv = e^{-st} \quad v = -\frac{1}{s} e^{-st} dt$

3/10

8. [10 marks] Use Laplace transforms to solve the initial value problem $y'' + y = \sqrt{2}\sin(\sqrt{2}t)$, $y(0) = 10$, $y'(0) = 0$ using a partial fraction expansion.

8/10

$y'' + y = \sqrt{2}\sin(\sqrt{2}t)$ $\quad y(0)=10$

$y'(0)=0$

$s^2 y(s) - s\,y(0)^{10} - y'(0)^{0} + y(s) = \frac{\sqrt{2}\sqrt{2}}{s^2+\sqrt{2}^2}$

$y(s)[s^2+1] - 10s = \frac{2}{s^2+2}$

$y(s)[s^2+1] = \frac{2+10s^3+20s}{s^2+2}$ ✓

$y(s) = \frac{2+10s^2+20s}{(s^2+2)(s^2+1)}$ $\qquad \frac{As+B}{(s^2+2)} + \frac{Cs+D}{(s^2+1)} = \frac{10s^2+20s+2}{(s^2+2)(s^2+1)}$

$(As+B)(s^2+1) + (Cs+D)(s^2+2) = 10s^2+20s+2$

$As^3 + As + Bs^2 + B + Cs^3 + 2Cs + Ds^2 + 2D = 10s^2+20s+2$

$y(s) = \frac{-20s}{(s^2+2)} + \frac{18}{(s^2+2)} + \frac{20}{s^2+1} - \frac{8}{(s^2+1)}$

$As^3 + Cs^3 = 0$ $\qquad A=-20$

$Bs^2 + Ds^2 = 10s^2$ $\qquad B=18$

$As + 2Cs = 20s$ $\qquad C=20$

$B + 2D = 2$ $\qquad D=-8$

$L\{y(s)\} = -20L\left\{\frac{s}{s^2+2}\right\} + 18L\left\{\frac{1}{(s^2+2)}\right\} + 20L\left\{\frac{1}{s^2+1}\right\}$

$-8L\left\{\frac{1}{s^2+1}\right\}$

$A=-C$

$-C+2C=20$

$C=20$

$B=10-D$

$10-D+2D=2$

$D=-8$

$y(s) = -20\cos\sqrt{2}t + 18\sin\sqrt{2}t + 20\sin t - 8\sin t$

$\boxed{y(s) = -20\cos\sqrt{2}t + 18\sin\sqrt{2}t + -12\sin t}$

9. [8 marks] Solve the initial value problem $y'' - y' = e^t \cos(t)$, with $y(0) = 0$, $y'(0) = 0$.

$$y'' - y' = e^t \cos(t) \qquad y(0) = 0$$
$$y'(0) = 0$$

$$s^2 Y(s) - s\cancel{y(0)}^{\,0} - \cancel{y'(0)}^{\,0} - sY(s) + \cancel{y(0)}^{\,0} = \frac{s-1}{(s-1)^2+1}$$

2

$$Y(s)[s^2 - s] = \frac{s-1}{(s-1)^2+1}$$

$$= \frac{s-1}{[(s-1)^2+1](s)(s-1)} = Y(s)$$

$$Y(s) = \frac{s-1}{[(s-1)^2+1]} \cdot \frac{\cancel{s-1}}{s(\cancel{s-1})}$$

$$Y(s) = e^t \cos t \cdot 1$$

$$\boxed{Y(s) = e^t \cos t}$$

10. [7 marks] Solve the initial value problem $y' + y = f(t)$, $y(0) = 0$ where $f(t) = \begin{cases} 0 & 0 \le t < 1, \\ 5 & t > 1. \end{cases}$.

$$sY(s) - \overset{0}{y(0)} + Y(s) = \int_1^\infty 5\,dt$$

$$Y(s)[s+1] =$$

0

/7

11. [10 marks] Assuming a power series solution $y = \sum_{n=0}^{\infty} c_n x^n$ for the differential equation $y'' - 2xy' + y = 0$ find the first three terms of two linearly independent solutions.

$$y'' - 2xy' + y = 0$$

$$\underbrace{\sum_{n=2}^{\infty} C_n(n)(n-1)x^{n-2}}_{k=n-2} - \underbrace{2x\sum_{n=1}^{\infty} C_n(n)x^{n-1}}_{k=n} + \underbrace{\sum_{n=0}^{\infty} C_n x^n}_{k=n}$$

$$\sum_{k=0}^{\infty} C_{k+2}(k+2)(k+1)x^k - 2\sum_{k=1}^{\infty} C_k(k)x^k + \sum_{k=0}^{\infty} C_k x^k$$

$$2C_2 + C_0 + \sum_{k=1}^{\infty}\left[C_{k+2}(k+2)(k+1) - 2C_k(k) + C_k\right]x^k$$

$$2C_2 + C_0 = 0$$

$$C_2 = -\frac{C_0}{2}$$

$$C_{k+2}(k+2)(k+1) + C_k[1-2k] = 0$$

$$C_{k+2} = \frac{C_k[-1+2k]}{(k+2)(k+1)}, \quad k=1,2,3$$

$$C_3 = \frac{-C_1}{6} \qquad C_4 = \frac{-1C_2}{4} \qquad C_5 = \frac{-1C_3}{4}$$

8/10

12. [10 marks] Using Laplace transform methods, **solve for** $y$ **only** in the following system: $\frac{d^2x}{dt^2}+x-y=0$ and $\frac{d^2y}{dt^2}+y-x=0$ with initial conditions $x(0)=0$, $x'(0)=-2$ and $y(0)=0$, $y'(0)=1$ .

$$s^2x(s)-s\cancel{x(0)}^{0}-\cancel{x'(0)}^{-2}+x(s)-y(s)=0$$
$$s^2y(s)-s\cancel{y(0)}^{0}-\cancel{y'(0)}^{1}+y(s)-x(s)=0$$

4/

$$x(s)[s^2+1]-y(s)+2=0$$
$$y(s)[s^2+1]-x(s)-1=0$$

$$y(s)=\frac{x(s)+1}{s^2+1}\to x(s)[s^2+1]^2-\frac{x(s)+1}{\cancel{s^2+1}}=-2(s^2+1)$$

$$x(s)=\frac{-2(s^2+1)-1}{(s^2+1)^2-1}$$

$$x(s)=\frac{-2(s^2+1)}{(s^2+1)^2-1}-\frac{1}{(s^2+1)^2-1}$$

$$x(s)=\frac{y(s)-2}{(s^2+1)}\Rightarrow y(s)[s^2+1]^2-\frac{y(s)+2}{\cancel{(s^2+1)}}=1(s^2+1)$$

$$y(s)=\frac{s^2+1}{(s^2+1)^2-1}-\frac{2}{(s^2+1)^2-1}$$

$$y(s)=\frac{(s^2+1)-2}{(s^2+1)^2-1}$$

2/

$$y(s)=\frac{s^2+1}{s^4+2s^2}-\frac{2}{s^4+2s^2}$$

$$y(s)=\frac{s^2+1}{s^2(s^2+2)}-\frac{2}{s^2(s^2+2)}$$

$$y(s)=\frac{\cancel{s^2}}{\cancel{s^2}(s^2+2)}+\frac{1}{s^2(s^2+2)}-\frac{2}{s^2(s^2+2)}$$

$$y(s)=\frac{1}{s^2+2}-\frac{1}{s^2(s^2+2)}$$

1/

$$L\{y(s)\}=\frac{1}{2}L\left\{\frac{2}{s^2+2}\right\}-L\left\{\frac{1}{s^2}\right\}-\frac{1}{2}L\left\{\frac{2}{s^2+2}\right\}$$

$$\boxed{y(s)=\frac{\sin\sqrt{2}t}{2}-t-\frac{\sin\sqrt{2}t}{2}}$$