homogeneous eqn = 0

Mid-term

2.2. seperable eq

when $\frac{dy}{dx} = g(x)h(y)$ & first order

step ① separate
② integrate both sides → (implicit soln)
③ isolate y → (explicit soln)

2.3. linear eqn of first order

eqn dep on y & y'

standard form $y' + p(x)y = q(x)$

Integrating factor

step ① write in standard form
② find integrating factor $IF = e^{\int p(x)dx}$
③ multiply everything by IF
④ note: the left simplifies to $(IF \cdot y)'$
⑤ integrate both sides
⑥ isolate y

method of variation

step ① solve HE (seperable) → gen sol HE
② plug into origional NHE
③ isolate c'
④ integrate
⑤ solve for C.
⑥ plug into GS.HE → gen sol NHE

guess

step ① guess PSNHE
② find GSHE (seperable)
③ add PSNHE + GSHE = GSNHE → gen sol NHE

2.4. Exact equations

$$\frac{\frac{dp}{dy} - \frac{dq}{dx}}{q}$$

$p(x,y)dx + q(x,y)dy = 0$

Criteria of exactness: $\frac{\delta p}{\delta y} = \frac{\delta q}{\delta x}$ & p & q are defined on all the domaine

step ① verify criteria of exactness
② integrate p in terms of x
③ derive that in terms of y
④ equate that to q
⑤ isolate c'
⑥ integrate to find C
⑦ plug C back into step 3
⑧ C = step 7 → soln

Making exact:

if $(p_y - q_x)/q$ is fct of $x \to \mu(x) = e^{\int \frac{p_y - q_x}{q} dx}$

if $(q_x - p_y)/p$ is fct of $y \to \mu(y) = e^{\int \frac{q_x - p_y}{p} dy}$

$\mu(pdx + qdy) = 0$

2.5. Bernoulli / substitution

if non- linear

Bernoulli eqn : $y' + p(x)y = q(x)y^{\alpha}$ $\alpha \neq 0$ or $1 \rightarrow$ non-linear

step ① write in Bernoulli form

② devide by y^{α}

③ x & ÷ y' by $\alpha - 1$

④ z = the y part of $p(x)$

⑤ it will simplify to $(\frac{1}{1-\alpha} z)' =$ wtv

⑥ now solve w z linear $\rightarrow$ sol for z

⑦ the replug y.

uv method

step ① prove by doing $*t$ & make sure same order

② then let $y = uv$

sub $u = \frac{y}{x}$ $y = ux$

③ the middle 2 terms $= 0$ & solve for v

④ replace v

⑤ integrate / solve

2.7 Linear Models

Growth & decay

$\frac{dx}{dt} - Kx = 0$

Newton's law of warming / cooling

$\frac{dT}{dt} = K(T - T_m)$ T_m: ambient temp

Solve by seperable eqn

Mid-term 2.

3.3 HE 2^{nd} order w/ cst coef $ay''+by'+Cy=0$

characteristic eqn $y''+\alpha y'+\beta=0 \to \lambda^2+\alpha\lambda+\beta=0$

① $D>0$

GSHE $= C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$

② $D=0$

GSHE $= C_1 e^{\lambda_1 x} + x C_2 e^{\lambda_1 x}$

③ $D<0$

GSHE $= C_1 e^{ax}\cos bx + C_2 e^{ax}\sin bx$

$\lambda = \frac{-\alpha \pm \sqrt{D}}{2} = \frac{-\alpha \pm \sqrt{-D}}{2} = \underbrace{\frac{-\alpha}{2}}_{a} \pm \underbrace{\frac{\sqrt{-D}}{2}}_{b} i$

3.4 Undetermined coef $ay''+by'+Cy=f(x)$

① guess

② undetermined coef

$P(x)$ 1) determine the order needed

2) write $y=Ax^n+Bx^{n-1}\ldots$ & the 2 derivatives

3) plug into orig eqn

4) solve for A, B, C...

5) GSNHE = PSNHE + GSHE

$e^{\alpha x}P_n(x)$ i) α not sol char eqn $\to y_p = e^{\alpha x}Q_n(x)$ → if α is not a soln to char eqn

ii) α simple root $y_p = xe^{\alpha x}Q_n(x)$ → if α is a soln to char eqn

iii) α double root $y_p = x^2e^{\alpha x}Q_n(x)$ → if α is only soln to char eqn

$P_n(x)\sin\beta x$ / $\cos$ i) βi $\not\in$ sol to char eqn $\to y_p = Q_n(x)\sin\beta x + R_n(x)\cos\beta x$

ii) βi sol to char eqn $\to y_p = xQ_n(x)\sin\beta x + xR_n(x)\cos\beta x$

if addition of multiple terms solve each seperatly then add them up.

3.5 Variation of parameters (to find PS to NHE) $ay'' + by' + cy = f(x)$

① find GSHE :

② $C_1' y_1 + C_2' y_2 = 0$

$C_1' y_1' + C_2' y_2' = f(x)$

③ solve (2) using cramer

④ integrate to find Cs

3.6 Cauchy-Euler Equation $ax^2y'' + bxy' + Cy = f(x)$

char eqn $ax^2y'' + bxy' + Cy = 0 \rightarrow a(\alpha)(\alpha-1) + b\alpha + c = 0 \rightarrow a\alpha^2 + (b-a)\alpha + c = 0$

① $D > 0$

$C_1 x^{\alpha_1} + C_2 x^{\alpha_2}$

② $D = 0$

$C_1 x^{\alpha} + C_2 x^{\alpha} \ln x$

③ $D < 0$

$C_1 x^{\beta} \cos(\gamma \ln x) + C_2 x^{\beta} \sin(\gamma \ln x)$ $\quad \alpha_1 = \overset{a}{\beta} + \overset{b}{i\gamma} \quad \alpha_2 = \beta - i\gamma$

$C_1 x^a \cos(b \ln x) + C_2 x^a \sin(b \ln x)$

*Cramer

$ax + by = \alpha$

$cx + dy = \beta$

$$x = \frac{\begin{vmatrix} \alpha & b \\ \beta & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{\alpha d - \beta b}{ad - bc}$$

$$y = \frac{\begin{vmatrix} a & \alpha \\ c & \beta \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{a\beta - c\alpha}{ad - bc}$$

*Zeros

$\dfrac{-\alpha \pm \sqrt{D}}{2}$ & $D = \alpha^2 - 4\beta$

$\dfrac{d}{dy} \ln y = 1$

3.7 Reduction of Order

Missing y

① let $y' = z$, $y'' = z'$, $y''' = z''$, etc

② solve

③ replug y's

Missing x

① let $y' = z(y)$, $y'' = z'(y)y' = z'(y)z$ ...

② solve seperable eqn

③ replug $\frac{dy}{dx}$

④ solve sep eqn again

3.8 linear models IVP

5.1 Power Series

step ① substitute $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

② replace n by k so that all n are the same
③ rewrite so all Σ start at the same number
④ each term needs to = 0
⑤ solve for a_ns

10.2 Homogeneous linear system $Y' = AY$

step ① Find eignevalues (λ) of A by $\det(A - \lambda I) = 0$

case 1: distinct eignevalues
② Find eignevectors (solve degenerate matrix for each λ) $\rightarrow V$
③ ANS = $c_1 e^{\lambda_1 t}\begin{pmatrix} \vdots \end{pmatrix} + c_2 e^{\lambda_2 t}\begin{pmatrix} \vdots \end{pmatrix} + c_3 e^{\lambda_3 t}\begin{pmatrix} \vdots \end{pmatrix} \cdots$

case 2: cmplx nbs, λ = cmplx
② Find eignevectors (solve degenerate matrix) (conjugate?) $\lambda = a + ib \rightarrow V$
③ $c_1 e^{\lambda t} V_1 + c_2 e^{\lambda t} V_2$
$c_1 e^{at} \cos t + i \sin t\left(\begin{pmatrix} a_{v_1} \\ a_{v_2} \end{pmatrix} + i\begin{pmatrix} b_{v_1} \\ b_{v_2} \end{pmatrix}\right) + c_2 \ldots$
ANS = $c_1 [e^{at} \cos t \begin{pmatrix} a_{v_1} \\ a_{v_2} \end{pmatrix} - \sin t \begin{pmatrix} b_{v_1} \\ b_{v_2} \end{pmatrix}] + c_2 [e^{at} \cos \begin{pmatrix} a_{v_1} \\ a_{v_2} \end{pmatrix} + \sin t \begin{pmatrix} b_{v_1} \\ b_{v_2} \end{pmatrix}]$

case 3: repeated λ = repeated
② find eignevectors
③ for repeated λ find eignevector normally & solve aug syst where insted of col of zeros, use eignevector you just found in order to find the last one.
④ ANS = $c_1 e^{\lambda_1 t}\begin{pmatrix} V_1 \end{pmatrix} + c_2 e^{\lambda_2 t}\begin{pmatrix} V_2 \end{pmatrix} + c_3 \left[t e^{\lambda_2 t}\begin{pmatrix} V_2 \end{pmatrix} + e^{\lambda_3 t}\begin{pmatrix} V_3 \end{pmatrix} \right]$

or if highly degenerate just make 2 vectors then norm ANS.

10.3 solution by diagonalization

① find eignevalues
② find eignevectors
③ matrix $A = (V\ V)$ & $Y = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \cdots \end{pmatrix}$ of $Y' = AY$

④ do multiplication to find Y' aka ANS & seperate according to cs to make vectors
ANS = $c_1\begin{pmatrix} \ \end{pmatrix} + c_2\begin{pmatrix} \ \end{pmatrix} + c_3\begin{pmatrix} \ \end{pmatrix}$

10.4 Nonhomogeneous linear systems $Y' = AY + F(t)$

GSNHS = GSHS + PSNHS

Variation of parameters

Steps ① solve for GSHS $\underbrace{C_1 e^{\lambda_1 t}(V_1)}_{Y_1} + \underbrace{C_2 e^{\lambda_2 t}(V_2)}_{Y_2}$

② write fundamental matrix $(\Phi) = (Y_1\ Y_2)$

③ PSNHE $= \Phi(t)\,c(t)$ where $c'(t) = \Phi^{-1}(t)\,F(t)$

so find Φ^{-1} (inverse) & multiply by $f(t)$ then integrate to find C.

My $C_1, C_2, C_3 \ldots$ of C vector now become my csts in PSNHE

undetermined coef. only work if $\lambda \neq 0$

steps ① solve for GSHS $C_1 e^{\lambda_1 t}(V_1) + C_2 e^{\lambda_2 t}(V_2)$

② solve syst of eqn $(A \vdots F(t)) \rightarrow (PP)$ & $PP(t)$

③ ANS: $C_1 e^{\lambda_1 t}(V_1) + C_2 \ldots + (PP) + PP(t)$

10.5 matrix exponential e^{At}

Steps: ① $sI - A$

② inverse

③ decompose into partial fraction

④ inverse again → ANS.

steps: ①

inverse of a matrix

3x3 and up $\left(A \,\middle|\, \begin{smallmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{smallmatrix}\right) \rightarrow \left(\begin{smallmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{smallmatrix} \,\middle|\, A^{-1}\right)$

2x2 diagonals switch, off diagonals get -ve & all ÷ by det so... $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = A^{-1}$

Variation of parameters

char eqn $ax^2y'' + bxy' + cy = 0 \rightarrow a\alpha(\alpha-1) + b\alpha + c = 0 \rightarrow a\alpha^2 + (b-a)\alpha + c = 0$

① $D > 0$

$C_1 x^{\alpha_1} + C_2 x^{\alpha_2}$

② $D = 0$

$C_1 x^{\alpha_0} + C_2 x^{\alpha_0} \ln x$

③ $D < 0$

$C_1 x^{\beta} \cos(\gamma \ln x) + C_2 x^{\beta} \sin(\gamma \ln x) \quad \alpha_1 = \beta + i\gamma \quad \alpha_2 = \beta - i\gamma$

NHE w/o cst coef

to find PSNHE

$$\left.\begin{aligned} C_1' y_1 + C_2' y_2 &= 0 \\ C_1' y_1' + C_2' y_2' &= \frac{F(x)}{ax^2} \end{aligned}\right\} \text{solve w/ cramer formula, then integrate to find } C_1 \,\&\, C_2$$

CRAMER

$ax + by = \alpha$

$cx + dy = \beta$

$$x = \frac{\begin{vmatrix} \alpha & b \\ \beta & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{\alpha d - \beta b}{ad - cb}$$

$$y = \frac{\begin{vmatrix} a & \alpha \\ c & \beta \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{a\beta - \alpha c}{ad - bc}$$

Cauchy-Euler Equation