MAT 1325: Calculus II and an Introduction to Analysis

Course Notes

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These course notes are adapted from my lecture notes from MAT1325 in Winter 2012. They were updated in Winter 2015.

These notes are not a substitute for attending class; reading math is less fun than participating in it in class. Beware of typos, errors, omissions and miscalculations. For more details, and a different perspective, read the corresponding section of our textbooks [S, T]. These are excellent books with very few typos and excellent, illustrated explanations and examples.

These notes have been developed from multiple sources, including principally [S, T] and some course notes of Barry Jessup's. They are intended exclusively for students registered in MAT1325/1725.

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Index of notation

$\mathbb{N}$ $= \{0, 1, 2, \cdots\}$, the natural numbers

$\mathbb{N}_0$ $= \mathbb{N}$, when we want to specifically include 0

$\mathbb{N}_+$ $= \{1, 2, \cdots\}$, when we want to specifically exclude 0

$\mathbb{Z}$ $= \{\cdots, -2, -1, 0, 1, 2, \cdots\}$, the integers

$\mathbb{Q}$ $= \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$ where $\frac{m}{n} = \frac{m'}{n'}$ when $mn' = m'n$, the rational numbers

$\mathbb{Q}^*$ $= \mathbb{Q} \setminus \{0\}$, the invertible rational numbers, that is, rational numbers excluding 0

$\mathbb{R}$ the real numbers

$\mathbb{R}^*$ $= \mathbb{R} \setminus \{0\}$, the real numbers, excluding 0

$\mathbb{C}$ $= \{x + iy \mid x, y \in \mathbb{R}\}$ where $i^2 = -1$, the complex numbers

$n!$: n factorial, defined for each $n \in \mathbb{N}_+$ by $n! = n(n-1)(n-2)\cdots 2 \cdot 1$, and $0! = 1$

$\binom{n}{k}$ $= \dfrac{n!}{(n-k)!k!}$, or "n choose k". This is the number of ways of drawing k marbles from a sack of n distinguishable marbles.

$\sum_{k=1}^{n} f(k)$ $= f(1) + f(2) + \cdots + f(n)$, a sum of terms, labeled and depending on an index k, where k takes all the integral values from 1 to n

$\forall$: for all

$\exists$: there exists

$A \Rightarrow B$: A implies B, if A is true then B is true

$A \Leftrightarrow B$: A is true if and only if B is true

iff : if and only if

$E \cup F$ $= \{x \mid x \in E \text{ ou } x \in F\}$, the union of two sets

$E \cap F$ $= \{x \mid x \in E \text{ et } x \in F\}$, the intersection of two sets

(a, b) $= \{x \in \mathbb{R} \mid a < x < b\}$, an open bounded interval

$[a, b]$ $= \{x \in \mathbb{R} \mid a \leq x \leq b\}$, a closed bounded interval

$[a, b)$ $= \{x \in \mathbb{R} \mid a \leq x < b\}$, a semi-open bounded interval

$(a, b]$ $= \{x \in \mathbb{R} \mid a < x \leq b\}$, a semi-open bounded interval

(a, ∞) $= \{x \in \mathbb{R} \mid x > a\}$, an open unbounded interval

$(-\infty, a]$ $= \{x \in \mathbb{R} \mid x \leq a\}$, a closed unbounded interval

$f\colon U \to \mathbb{R}$, a function defined on the domain $U \subseteq \mathbb{R}$ and taking values in $\mathbb{R}$

$\mathbb{R}^2$ $= \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$ ordered pairs of real numbers

Chapter 1

Introduction: Real numbers

The real numbers are a bit like your parents[1]. You've known them forever, you're very familiar with them, but then you might learn something new about them that makes your jaw drop and challenges all your assumptions.

Calculus I was all about functions on the real numbers, so we start by looking at a few of our assumptions about numbers.

1.1 What are numbers?

1.1.1 Some thoughts about numbers

There are many different sets we call "numbers":

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

where

- $\mathbb{N} = \{0, 1, 2, \cdots\}$ (natural numbers),
- $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ (integers),
- $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$ where we say $\frac{m}{n} = \frac{m'}{n'}$ whenever $mn' = m'n$ (rational numbers),
- $\mathbb{R} = \mathbb{Q} \cup \{?\}$ (real numbers),
- $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$ where $i^2 = -1$ (complex numbers).

We know that $\mathbb{R}$ is bigger than $\mathbb{Q}$; the Pythagoreans knew this.

Theorem 1.1. *The "number" $\sqrt{2}$ is irrational, that is, it is not in $\mathbb{Q}$.*

[1]With thanks to Barry Jessup, for providing the apt analogy.

Proof. We argue by contradiction. Suppose instead that $\sqrt{2} \in \mathbb{Q}$. Then there are integers m and n, with no common factors other than 1[2], such that

$$\sqrt{2} = \frac{m}{n}.$$

Now square both sides, to get

$$2 = \frac{m^2}{n^2}$$

which means

$$2n^2 = m^2.$$

So m^2 is even. The square of an odd integer is odd and the square of an even integer is even. (Exercise) Therefore m is even. Write $m = 2k$ for some integer k; then in fact $m^2 = 4k^2$. Putting this into the equation above yields

$$2n^2 = 4k^2.$$

Dividing by 2 gives $n^2 = 2k^2$, so again n^2 is even, implying n is even. So m and n are both even, that is, they have 2 as a common factor.

But this is absurd, a contradiction: we said that m and n had no common factors besides 1.

> When we have a contradiction, it means we did something wrong; but the only thing we did wrong here was to suppose that $\sqrt{2} \in \mathbb{Q}$.

Consequently, we conclude that our initial hypothesis ($\sqrt{2} \in \mathbb{Q}$) was false; whence its negation ($\sqrt{2} \notin \mathbb{Q}$) is true. □

> This was a proof by contradiction, also called reductio ad absurdum: we argued that if the theorem were false, then a logical contradiction would occur, i.e. something simple would be forced to be both true and false. So the theorem cannot be false, whence it is true.

That said, you don't get all of $\mathbb{R}$ just by adding things like $\sqrt{2}$ (that is, roots of polynomial equations) to $\mathbb{Q}$; $\mathbb{R}$ is much bigger than that. So how can we describe it?

Our "working definition" of the real numbers, the one we are familiar with, is:

$$\mathbb{R} = \{\text{all decimal expansions, infinite and finite}\}.$$

By performing long division, "until infinity" if necessary, we see that every rational number has a decimal expansion, so $\mathbb{Q} \subset \mathbb{R}$. Also

$$\sqrt{2} = 1.411421\ldots, \quad \pi = 3.14159\ldots$$

are other elements of $\mathbb{R}$ (except the ... means we haven't fully specified these numbers; what number do we mean? Hmmmm... food for thought.). There are some little problems with this definition, though.

[2]Two integers with no common factors besides 1 (or -1) are called *relatively prime*.

Lemma 1.2. *We have $0.\overline{9} = 1$, where $\overline{9}$ means the infinitely repeating pattern $99999\cdots$.*

Proof. Let $x = 0.\overline{9} = 0.9999999\cdots$. Then $10x = 9.99999\cdots$ and so $10x - x = 9$. But this means $9x = 9$, whence $x = 1$. □

> This was a direct proof. We started with $x =$ left side, did some math, and ended up with $x =$ right side.

This kind of argument applies to any real number whose decimal expansion ends in $\overline{9}$.

What is even weirder: how do we add two real numbers? For example, how do we add

$$\sqrt{2} = 1.4142135623730950488016887\cdots$$

and

$$\pi = 3.14159265358979323846264338327 9\cdots \quad ?$$

Our algorithm for adding decimal numbers doesn't work since we can't "start" at the rightmost digit, because there is no last digit. If we truncate we can add, but the last digit of the sum (or more!) could be wrong. Multiplication is just as impossible. (What we really need is the notion of a *sequence* of rational numbers that *converge* to the given real number; then this problem disappears. (Next week.))

So our problem is that we know everything and nothing about the real numbers; we can feel uncertain about things because what we know are a huge collection of so-called "facts" about the real numbers, without the certainty that they are all correct that comes from logical proof. Historically, the answer to this was the axiomatic development of the real numbers, which we move towards next.

1.1.2 Properties of $\mathbb{Q}$ and $\mathbb{R}$; Axioms for a field

Let's spell out the key properties of $\mathbb{Q}$ and see how these properties distinguish them from some of the other number systems. These are properties we know hold true for $\mathbb{R}$ as well, up to the glitch we have of defining addition and multiplication for $\mathbb{R}$.

Axioms for Addition

There is an operation called *addition* which is a map which takes two numbers x and y and outputs a number called $x + y$, satisfying the following properties (for any numbers x, y, z):

A1 $x + y = y + x$ (commutativity)

A2 $(x + y) + z = x + (y + z)$ (associativity)

A3 $0 + x = x + 0 = x$ for all numbers x (the existence of a neutral element)

A4 For every number x there exists a number y such that $x + y = 0$ (existence of an additive inverse). We write $y = -x$. (See exercises)

Remark 1.3. Note that A1-A4 hold if "numbers" are $\mathbb{Z}$, or $\mathbb{Q}$, or $\mathbb{R}$; but that A4 does NOT hold if "numbers" are $\mathbb{N}$. So already some sets of numbers are better than others.

Remark 1.4. If you have taken MAT1341, then you will recognize that A1-A4 are also part of the axioms for a vector space. That is, if you replace "number" with "vector" (or "function", even) then A1-A4 hold. These are very general axioms that we expect when we use the symbol +.

Remark 1.5. The element 0, if it exists, is unique; also when x has an additive inverse, then there's only one. See the exercises.

The associativity and commutativity of addition mean we can add any (finite) number of elements any way we like. Recall that we abbreviate

$$a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

for any numbers $a_1, \ldots, a_n$.

Axioms for Multiplication (stated for $\mathbb{R}$)

There is also an operation called *multiplication* which is a map which takes as input $x, y \in \mathbb{R}$ and gives as output $xy \in \mathbb{R}$, and which satisfies the following properties (for any $x, y, z \in \mathbb{R}$):

M1 $xy = yx$ (commutativity)

M2 $(xy)z = x(yz)$ (associativity)

M3 $1x = x1 = x$ for all $x \in \mathbb{R}$ (the existence of a neutral element)

M4 For every $x \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$, there exists a $y \in \mathbb{R}^*$ such that $xy = 1$ (existence of a multiplicative inverse). We write $y = x^{-1}$ or $y = 1/x$.

Moreover, addition and multiplication relate to one another in the following way:

D1 For any $x, y, z \in \mathbb{Q}$, we have $x(y + z) = xy + xz$ (distributivity)

Remark 1.6. We could put $\mathbb{Q}$ in place of $\mathbb{R}$, and these axioms would hold. But if we put $\mathbb{Z}$ in place of $\mathbb{R}$, then axiom M4 fails (and the rest hold). So $\mathbb{Z}$ does not satisfy as many axioms as $\mathbb{R}$ or $\mathbb{Q}$ do.

Remark 1.7. You can't multiply vectors to produce another vector, usually, so these axioms are not part of the axioms for a vector space. In $\mathbb{R}^3$, there is a cross-product, denoted $x \times y$ which satisfies D1 — but it doesn't satisfy M1 or M2 or M3 (so M4 doesn't even make sense to ask). This is why no one calls it a multiplication.

The collection of axioms A1-A4, M1-M4 and D1 describe a *field*; so $\mathbb{Q}$ and $\mathbb{R}$ are fields.[3]

[3] The complex numbers $\mathbb{C}$ are also a field, as is the set $\{0, 1\}$ with binary addition $1 + 1 = 0$.

Fact: these axioms are enough to ensure that familiar identities hold, like $(x+y)^2 = x^2+2xy+y^2$ or $-x = (-1)x$ or $0x = 0$. (See the exercises.)

Example 1.8. (Optional) Show that the multiplicative identity (that is, the element satisfying M3) in a field is unique.

Proof: Suppose that there were two nonzero elements, call them n and m, which satisfied the property that for all nonzero x, we had $xn = nx = x$ and $mx = xm = x$. So in particular, we'd have $nm = n$ since m is nonzero and n is a multiplicative identity; and we'd have $nm = m$ since n is nonzero and m is a multiplicative identity. Hence $n = m$, that is, there is only one multiplicative identity in a field, and without loss of generality, we denote it by the symbol 1.□

> This is the typical way you show something is unique: you suppose you have two instances and deduce they must be equal. Ergo, there cannot be two distinct instances, and so if it exists, it is unique.

Example 1.9. (Optional) Show that the multiplicative inverse of a nonzero element (that is, an element satisfying M4) in a field is unique.

Proof: Let x be a nonzero element of the field and suppose y, z are nonzero elements which both satisfying the conditions of axiom M4, that is, such that we have the equalities $xy = yx = 1$ and $xz = zx = 1$. Then consider the product yxz. By axiom M2, we can expand it in two different ways:

$$yxz = (yx)z = 1z = z \quad \text{and} \quad yxz = y(xz) = y1 = y$$

Hence $y = z$ and so there's only one multiplicative inverse in a field. Without loss of generality, we denote it x^{-1}. □

Example 1.10. (Optional) Show that $(bd)^{-1} = b^{-1}d^{-1}$.

Proof (direct): We need to show that bd times $b^{-1}d^{-1}$ is 1 because that is the definition of the expression "$(bd)^{-1}$". So we have:

$$\begin{aligned}(bd)(b^{-1}d^{-1}) &= (bb^{-1})(dd^{-1}) \quad (M1, M2)\\ &= 1 \cdot 1 \quad (M4)\\ &= 1 \quad (M3)\end{aligned}$$

as required. □

Example 1.11. (Optional) Show that $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ for any $b, d \neq 0$. (Note that here we are using the abbreviation $\frac{a}{b}$ for ab^{-1}.)

Proof (direct):

$$\begin{aligned}
\frac{a}{b} + \frac{c}{d} &= ab^{-1} + cd^{-1} \quad \text{(definition)} \\
&= 1ab^{-1} + 1cd^{-1} \quad (M3) \\
&= dd^{-1}ab^{-1} + bb^{-1}cd^{-1} \quad (M4) \\
&= b^{-1}d^{-1}ad + b^{-1}d^{-1}bc \quad (M1, M2) \\
&= b^{-1}d^{-1}(ad + bc) \quad (D1) \\
&= (bd)^{-1}(ad + bc) \quad \text{(previous example)} \\
&= \frac{ad + bc}{bd} \quad \text{(definition)}\square
\end{aligned}$$

We can also define, for any $x \in \mathbb{R}$, and $n \in \mathbb{N}$ with $n \geq 1$

$$x^n = x \cdot x \cdot \ldots \cdot x \ (n \text{ times})$$

and if $x \neq 0$, we can define $x^0 = 1$ and

$$x^{-n} = (x^{-1})^n.$$

However, 0^0 is undefined (just like 0^{-1} is undefined). It is often convenient for us to take $0^0 = 1$. See the exercises.

Lemma 1.12. *For any $x, y \neq 0$ and $m, n \in \mathbb{Z}$ we have*

(a) $(x^n)^{-1} = (x^{-1})^n$

(b) $(xy)^n = x^n y^n$

(c) $x^{m+n} = x^m x^n$

(d) $x^{mn} = (x^m)^n$

Note that in general $(x + y)^n \neq x^n + y^n$.

Proof. (Optional) (a) If $n > 0$ then note that $x^n(x^{-1})^n = (xx^{-1})^n$ by commutativity and associativity; and this equals $1^n = 1$. Therefore $(x^n)^{-1} = (x^{-1})^n$. If $n = 0$ then both sides equal 1 by definition. If $n < 0$, then $x^n = (x^{-1})^{-n}$ and $(x^{-1})^n = ((x^{-1})^{-1})^{-n}$. Since $(x^{-1})^{-1} = x$ (exercise), we have $(x^n)(x^{-1})^n = (x^{-1})^{-n}x^{-n}$. Since $-n > 0$, this can be rearranged by commutativity and associativity to give $(x^{-1}x)^{-n} = 1$ as above; whence x^n is the multiplicative inverse of $(x^{-1})^n$ in this case as well.

This was a proof by cases: we dealt with $n < 0$, $n = 0$ and $n > 0$ separately. Since these three cases cover all possibilities, we are done.

(b)-(d) are left as exercises. For $m, n \in \mathbb{N}$, the results are obtained by counting; for negative n, m, one must first apply the definition of a negative exponent (as the corresponding positive power of x^{-1}). $\square$

Remark 1.13. It is not true that we can let the exponent n be an element of the field. Besides problems like $(-1)^{1/2}$ being undefined, we've already shown that $2^{1/2} \notin \mathbb{Q}$ even though 2 and $\frac{1}{2}$ clearly are. So this isn't the same kind of operation as addition and multiplication; see also the exercises.

1.1.3 Exercises

Now that you've read some interesting math, it's time to learn it. Mathematics is learned by **doing** — putting abstract new concepts into practice so that they become concrete skills you can apply in future.

Here is a selection of exercises based on what we covered in the first lecture. Some of these are included on the list of problems you may discuss in the DGD.

The "challenge problems" are beyond the scope of this course but included for your personal interest. Choose some of these exercises after each class as a way to gauge your understanding of the material; also do them in preparation for the midterm and final exam.

1. An integer is defined to be even if it is divisible by 2; so every even number m can be written as $m = 2k$ for some integer k. If m is not even it is called odd, and so it can be written as $m = 2k + 1$ for some integer k. Use these characterizations of even and odd integers to prove that the square of an even integer is even and the square of an odd integer is odd.
2. Show that every infinite repeating decimal expansion represents a rational number. (Hint: multiply x by 10^n for some n, such that the difference $10^n x - x$ is an integer.)
3. Give another proof that $1 = 0.\overline{9}$, using $\frac{1}{3}$.
4. (Challenge problem: a famous classic argument) Show that the number of rational numbers is countable, that is, there exists an algorithm for enumerating all rational numbers. Or, do this only for the positive rationals. Hint: can you find a path through the set of all pairs of positive integers (in the xy plane, for example)?
5. (Challenge problem: a famous classic argument, due to Cantor) Show that the set $\{x \in \mathbb{R} \mid 0 \leq x < 1\}$ is not countable, that is, there does not exist an algorithm which can enumerate all numbers in this set. Hint: if there were an algorithm, you could make a list of all numbers; can you use the list to create a number that cannot possibly be on the list?
6. Prove that if z is an element of a field that satisfies the same property as 0 in A3 (namely, for every $x \in \mathbb{R}$, $x + z = z + x = x$) then in fact $z = 0$. (You can use A4!) In other words, we are showing that the neutral element for addition in a field is unique. Do the same for M3: that is, show that the neutral element for multiplication is unique.
7. Prove that if x is an element of a field then its additive inverse is unique, that is, if y, z both satisfy the condition of axiom A4, then necessarily, $y = z$. Use the axioms — justify every move via an axiom. The same proof works for the multiplicative case.
8. Prove that $0x = 0$ for any $x \in \mathbb{Q}$. Hint: you are describing an interplay between addition and multiplication, so axiom D1 must be used.

9. Prove that $(-1)x = -x$, that is, the additive inverse of x is obtained by multiplying x by $-1 \in \mathbb{Q}$. Hint: you are describing an interplay between addition and multiplication, so axiom D1 must be used.

10. Prove that in a field, if $xy = 0$, then either $x = 0$ or $y = 0$. (Hint: to prove "*this* or *that*", a good logical technique to use is the following: "If *this* is true, then we're done. So assume *this* is NOT true, and let's show that this forces *that* to be true." Otherwise, your proof would be like trying to hit a moving target.)

11. Prove the following properties, for any elements a, b, c, d of a field (which are assumed to be nonzero as appropriate in each statement):
    (a) $1/(1/a) = a$ (Notice, by the way, that $1/(1/a)$ is not the same thing as $(1/1)/a$; it's multiplication that is associative, not division. Hence our preference for writing a^{-1} instead of $1/a$...)
    (b) $(a/b)/(c/d) = (a/b)(d/c)$
    (c) $(a/b)(c/d) = (ac)/(bd)$
    (d) $(a/c) + (b/c) = (a + b)/c$
    (e) $-(a/b) = (-a)/b$

12. Prove using the axioms that $(x + y)^2 = x^2 + 2xy + y^2$ for $x, y \in \mathbb{R}$.

13. Prove Lemma 1.12.

14. We like to take $0^0 = 1$, but we could just as easily have chosen $0^0 = 0$, which is why 0^0 is "undefined." Normally, there is only one choice that is consistent with all axioms. Show that both choices are consistent with the laws of exponents.

15. Show that division is not an associative operation, that is, does not satisfy the corresponding axiom to A2 or M2.

16. Show that exponentiation (the map which associates to $x, y \in \mathbb{N}$ the value x^y) is not commutative and not associative.

1.2 Ordered fields

There are many fields (really, really, really many), but what starts to distinguish $\mathbb{Q}$ and $\mathbb{R}$ is that they are ordered fields, defined as follows.

1.2.1 Order relation

The real numbers are ordered by a relation $<$ which satisfies the following properties (for any $x, y, z \in \mathbb{R}$):

R1 (Total order) For any $x, y \in \mathbb{R}$, exactly one of the following holds:

$$x < y \quad \text{or} \quad x = y \quad \text{or} \quad x > y,$$

(where $x > y$ means $y < x$).

R2 If $x < y$ and $y < z$ then $x < z$ (transitivity).

R3 If $x < y$ then $x + z < y + z$ (preserved under addition).

R4 If $x < y$ and $z > 0$ then $zx < zy$ (preserved under multiplication by *positive* numbers).

As usual, we write $x \leq y$ to mean "either $x = y$ or $x < y$"; and we define the positive numbers to be

$$\{x \mid x > 0\}$$

and the negative numbers as $\{x \mid x < 0\}$. We use the terms "nonnegative" and "nonpositive" if we want to include 0.

A field satisfying axioms R1-R5 is called an *ordered field*.

Example 1.14. Both $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields but $\mathbb{C}$ is not (see the exercises).

Lemma 1.15. *The following statements are true in an ordered field. For any x, y, z we have:*

(a) If $x < y$ and $z < 0$ then $zx > zy$ (multiplying by negative numbers reverses the inequality).

(b) $1 > 0$

(c) If $x > 0$ then $-x < 0$ and $x^{-1} > 0$.

(d) If $x < 0$ then $-x > 0$ and $x^{-1} < 0$.

(e) If $x > 1$ then $x^{-1} < 1$.

Proof. (Optional) (a) Suppose $z < 0$. Then adding $-z$ to both sides by R3 yields $0 < -z$. We may therefore by R4 multiply both sides of $x < y$ by $-z$ to get $(-z)x < (-z)y$. By an exercise, we know that $-z = (-1)z$ and therefore we can apply associativity to get $-zx < -zy$. Finally, adding $zx + zy$ to both sides via R3 yields $zy < zx$, as required.

(b) By definition, $1 \neq 0$ so by R1, either $1 > 0$ or $1 < 0$. Let us assume to the contrary that $1 < 0$. Then by (a), multiplying the inequality $1 < 0$ by 1 should reverse the inequality, yielding $1 > 0$. This is a contradiction of R1. Therefore $1 > 0$.

(c) Suppose $x > 0$. Adding $-x$ to both sides using R3 yields $0 > -x$. We know $x^{-1} \neq 0$; if $x^{-1} < 0$ then multiplying $x > 0$ by the negative element x^{-1} would reverse the inequality, giving $x^{-1}x < x^{-1}0$; but $x^{-1}x = 1$ and $x^{-1}0 = 0$ (by an exercise), so this says $1 < 0$, which contradicts (b). Therefore $x^{-1} > 0$.

(d) (exercise)

(e) If $x > 1$ then since by (b) $1 > 0$, we have by R2 than $x > 0$. Therefore $x^{-1} > 0$ by (c). Multiplying $x > 1$ by x^{-1} then yields, by R4, the desired inequality. □

Example 1.16. Solve $x^3 + 3x < -4x^2$.

Solution: By R3 we can add $4x^2$ to both sides to get the equivalent expression

$$x^3 + 4x^2 + 3x < 0, \quad \text{or} \quad x(x+1)(x+3) < 0.$$

One can prove (using R4 and Lemma 1.15(a)) that a product of three terms is negative if and only if an odd number of terms is negative. Noting that $x < a$ is equivalent to $x - a < 0$ (and similarly for $>$ and $=$), and that the roots of the cubic are $-3 < -1 < 0$, we see that this reduces to asking that x is less than an odd number of roots. Thus the solution is all values x such that $x < -3$ or $-1 < x < 0$. We write this in interval notation as

$$(-\infty, -3) \cup (-1, 0)$$

to represent the union of these two intervals.

> Danger: it would be INCORRECT to divide through by x to simplify the above inequality. If you "remove" the x, you get
>
> $$(x+1)(x+3) < 0$$
>
> which has solution the interval $(-3, -1)$, which is clearly incorrect. The reason we get the wrong answer: "cancelling x" is not a magical operation, but rather accomplished by multiplying by x^{-1}. Lemma 1.15(a) says that multiplying by a negative reverses the inequality. Therefore the argument provides correct answers only for $x > 0$ (as we can see).

Example 1.17. Find all x such that $\frac{1}{3-x} < 2$.

Bad solution:

$$\frac{1}{3-x} < 2 \quad \Rightarrow \quad 1 < 2(3-x), \quad \Rightarrow 1 < 6 - 2x \Rightarrow 2x < 5 \Rightarrow x < \frac{5}{2}. \tag{1.1}$$

A correct solution: If $x < 3$ then $3 - x > 0$ so

$$\frac{1}{3-x} < 2 \quad \Rightarrow \quad 1 < 2(3-x), \quad \Rightarrow 1 < 6 - 2x \Rightarrow 2x < 5 \Rightarrow x < \frac{5}{2}.$$

Thus in this case the inequality holds iff $x < \frac{5}{2}$.

If $x > 3$ then $3 - x < 0$ so instead we get

$$\frac{1}{3-x} < 2 \quad \Rightarrow \quad 1 > 2(3-x), \quad \Rightarrow 1 > 6 - 2x \Rightarrow 2x > 5 \Rightarrow x > \frac{5}{2}.$$

But this is true of all x in this case; thus the inequality always holds in this case. This gives $x > 3$.

So the final answer: $x \in (-\infty, \frac{5}{2}) \cup (3, \infty)$.

> A better solution:
>
> $$\frac{1}{3-x} < 2 \quad \Leftrightarrow \quad 2 - \frac{1}{3-x} > 0 \quad \Leftrightarrow \quad \frac{6 - 2x - 1}{3-x} > 0 \quad \Leftrightarrow \quad \frac{5-2x}{3-x} > 0$$
>
> which holds iff either both $5 - 2x > 0$ and $3 - x > 0$, or both $5 - 2x < 0$ and $3 - x < 0$. The roots are $\frac{5}{2}$ and 3, so solving these inequalities yields the answer.

Example 1.18. Solve $x^3 \leq x$.

Solution: If $x = 0$ the inequality holds.

If $x > 0$, then it is equivalent to $x^2 \leq 1$ (by multiplying by the positive number x^{-1}). Since for positive y, $y \leq 1$ iff $y^2 \leq 1$ (exercise), we conclude $0 < x \leq 1$.

On the other hand, if $x < 0$, then this is equivalent to $x^2 \geq 1$. As above, we can conclude that this is true only if $x \leq -1$.

Our final solution: $x \in (-\infty, -1] \cup [0, 1]$.

1.2.2 Absolute value and the triangle inequality

Given an order, and thus a notion of positive and negative numbers, we can define the absolute value on $\mathbb{R}$ or $\mathbb{Q}$ as follows.

Definition 1.19. If $x \in \mathbb{R}$, then

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0. \end{cases}$$

Note that by Lemma 1.15(d), $|x| \geq 0$ for all $x \in \mathbb{R}$.

This defines a function $|\cdot|\colon \mathbb{R} \to \mathbb{R}^+$ where $\mathbb{R}_0^+ = \{x \in \mathbb{R} \mid x \geq 0\}$, which we call the *absolute value*, or *modulus*, or *norm*.

Lemma 1.20. *Let $r > 0$ and $a \in \mathbb{R}$. Then for every $x \in \mathbb{R}$ we have*

(a) $|x| < r$ if and only if $x \in (-r, r)$;

(b) $|x| \leq r$ if and only if $x \in [-r, r]$;

(c) $|x - a| < r$ if and only if $x \in (a - r, a + r)$;

(d) $|x - a| \leq r$ if and only if $x \in [a - r, a + r]$.

Proof. (a) We need to prove two implications: that if $|x| < a$ then $-a < x < a$; and that if $-a < x < a$ then $|x| < a$.

Suppose first that $|x| < a$. If $x \geq 0$, then $|x| = x$ so we have $x < a$; and since $-a < 0 \leq x$, we see that $-a < x < a$. On the other hand, if $x \leq 0$, then $|x| = -x$, so we have $-x = |x| < a$ which implies, by multiplying through by -1, that $x > -a$. Again, since $a > 0 \geq x$ we conclude $-a < x < a$.

Now suppose that $-a < x < a$. Multiplying through by -1 yields $a > -x > -a$, so that both x and $-x$ are in the interval $(-a, a)$. But $|x|$ is equal to either x or $-x$, so it follows that $|x| \in (-a, a)$, which in particular implies $|x| < a$.

(b) We know from (a) that $|x| < r$ if and only if $x \in (-r, r)$. We also know that $|x| = r$ if and only if either $x = r$ or $x = -r$. Therefore the result holds.

(c) From (a), we know that $|y| < r$ if and only if $-r < y < r$. Set $y = x - a$; then we have $|x-a| < r$ if and only if $-r < x - a < r$. But, adding a to all sides, we see that $-r < x - a < r$ if and only if $a - r < x < a + r$, which is the same as $x < (a - r, a + r)$.

(d) exercise. □

Theorem 1.21 (The triangle inequality)**.** *For all $x, y \in \mathbb{R}$, we have*

$$|x + y| \leq |x| + |y|$$

Proof. Note that if $a \geq 0$, then $|a| = a$; and if $a < 0$, then $a < 0 < -a = |a|$. Thus in all cases, we have

$$a \leq |a|.$$

To prove the inequality, we consider two cases: $x + y \geq 0$ or $x + y < 0$.

If $x + y \geq 0$, then we have

$$|x + y| = x + y \leq |x| + |y|.$$

If $x + y < 0$, then we have

$$|x + y| = -(x + y) = (-x) + (-y) \leq |-x| + |-y| = |x| + |y|.$$

Thus the inequality holds for all x, y. □

Example 1.22. Show that $|x| \leq 3$ implies $|x - 5| \leq 8$.

Draw a number line to agree that this is true. This is useful for keeping your thoughts straight, and help you to decide if your steps are leading in the right direction.

Proof: we start with the hypothesis: let $x \in \mathbb{R}$ be such that $|x| \leq 3$.

$$\begin{aligned} |x - 5| &= |x + (-5)| \\ &\leq |x| + |-5| \quad \text{by the triangle inequality} \\ &= |x| + 5 \\ &\leq 3 + 5 \quad \text{by hypothesis} \\ &= 8. \end{aligned}$$

Remark 1.23. However, we cannot conclude the converse! That is, it is NOT true that $|x-5| \leq 8$ implies $|x| \leq 3$ (as you can readily see).

Why not? Because the above argument is not reversible: The fact that $|x - 5| \leq 8$ and that $|x - 5| \leq |x| + 5$ neither implies $|x| + 5 \leq 8$, nor $8 \leq |x| + 5$. (Find some examples!)

Beware the common trap: knowing $a < b$ and $a < c$ does not imply that $b < c$, even if you really, really, really want it to be true.

Example 1.24. Prove that $\forall x, y, z \in \mathbb{R}$, $|x - z| \leq |x - y| + |y - z|$. This is the origin of the term "triangle inequality", if you replace $\mathbb{R}$ with $\mathbb{R}^2$.

Solution: Let $u = x - y$ and $v = y - z$. Then $u + v = x - z$. Therefore this inequality follows from the triangle inequality applied to u and v, that is, $|u + v| \leq |u| + |v|$.

1.2.3 Exercises

1. Which of the following statements are true? (Recall: "sometimes true and sometimes false" logically means "false".) For those that are false, find related statements that are true.

   (a) for all $x, y \in \mathbb{R}$, if $x > y$ then $x^2 > y^2$.

   (b) for all $x, y \in \mathbb{R}$, $(x + y)^2 = x^2 + y^2$.

   (c) for all $x, y \in \mathbb{R}$, $\sqrt{x^2 + y^2} = |x| + |y|$.

   (d) for all $x, y \in \mathbb{R}$, if $x > y$ then $|x| > |y|$.

   (e) for all $x, y \in \mathbb{R}$, if $x > y$ then $|y - x| < |y|$.

   (f) for all $a, b \in \mathbb{R}$, such that $a > 0$ and $b > 0$, $\sqrt{a + b} = \sqrt{a} + \sqrt{b}$.

   (g) for all $x, y \in \mathbb{R}$, and any $n \in \mathbb{N}$, $(x + y)^n = x^n + y^n$.

2. Verify that the set $\{0, 1\}$ with addition and multiplication modulo 2, is a field.

3. Show that $\{0, 1\}$ is not an ordered field. Hint: by R2, $0 \neq 1$ so either $0 < 1$ or $1 < 0$. Show that either case leads to a paradox, that is, a failure of one of the axioms.

4. Show that $\mathbb{C}$ is not an ordered field as follows (that is, fill in the details and required justification). If $i > 0$ then $i^2 > 0$ which is a contradiction (why?). If $i < 0$ then $i^2 > 0$, again a contradiction. Since $i \neq 0$, we see that $\mathbb{C}$ cannot be ordered.

5. Prove the following properties of an ordered field. For any x, y, a, b we have:

   (a) $x < y$ iff $y - x > 0$.

   (b) $-1 < 0$

   (c) If $x < y$ and $a \leq b$ then $x + a < y + b$.

   (d) If $0 < x < y$ and $0 < a < b$ then $ax < by$.

   (e) Lemma 1.15 (d)

   (f) The product of positive numbers is positive, the product of two negative numbers is positive, and the product of a positive number and a negative number is negative.

   (g) The product of three numbers is negative iff an odd number of the terms are negative.

6. Prove that if $x > 0$ then $0 < x/2 < x$. Hint: show $2 > 1$, and then use Lemma 1.15(e) to see $1/2 < 1$; multiply by x.

7. Prove that for all $x \in \mathbb{R}$, $|x| = |-x|$.

8. Prove that $|x| = 0$ if and only if $x = 0$.

9. Suppose that $a \in \mathbb{R}$ and for every $\varepsilon > 0$ we have $|a| < \varepsilon$. Prove that $a = 0$.

10. Show that if $a \in \mathbb{R}$ and $\varepsilon > 0$ then $\{x \in \mathbb{R} \mid |x - a| < \varepsilon\} = \{x \in \mathbb{R} \mid a - \varepsilon < x < a + \varepsilon\}$. This latter set is an interval, denoted $(a - \varepsilon, a + \varepsilon)$.

11. Show that for any x, y $|x + y| \geq |x| - |y|$.

12. Show that for any x, y, $|x - y| \geq ||x| - |y||$.

13. Show that for any $x, y \in \mathbb{R}$, $|xy| = |x|\ |y|$.

1.3 The supremum of a set

We saw that the axioms A1-A4, M1-M4, D1, R1-R5 apply to both the rational numbers and, from our general knowledge, the real numbers[4] (There are still some glitches for us to work through, however, such as how to actually add and multiply real numbers.) There is one more axiom to add that will distinguish the real numbers as more special than the rational numbers: the completeness axiom. To state it we need to define the supremum of a set, which is a sophisticated version of the maximal element that makes things interesting.

Consider the variety of possible sets of real numbers.

$$\begin{aligned} S_1 &= [1,5] & S_2 &= (1,5) \\ S_3 &= (43,\infty) & S_4 &= \{1,5\} \\ S_5 &= (0,2)\cup[4,6]\cup\{11\} & S_6 &= \{\frac{1}{n} \mid n\in\mathbb{N}_+\} \\ S_7 &= \mathbb{N} & S_8 &= \{1-\frac{1}{n^2} \mid n\in\mathbb{N}_+\} \end{aligned}$$

There are also crazier types of sets:

$$\begin{aligned} S_9 &= \{x\in[1,3] \mid \text{its decimal expansion consists only of 1s and 2s}\} \\ S_{10} &= \{\frac{p-1}{p} \mid p \text{ is a prime}\}, \end{aligned}$$

and then some REALLY horrible sets:

$$\begin{aligned} S &= \{\sin(x) \mid x\in\mathbb{R}\setminus\mathbb{Q}\} \\ S &= \{x\in(0,1) \mid x \text{ is a sum of distinct fractions of the form } \frac{1}{m}\} \\ S &= \{x\in(0,1) \mid x \text{ is the root of some } p(x)\in\mathbb{Z}[x]\} \\ S &= \text{a specific infinite list of real numbers between 0 and 1} \\ S &= \text{a Cantor set.} \end{aligned}$$

1.3.1 Does every set of real numbers contain a maximal element?

The answer is obviously NO; some sets "go off to infinity", containing bigger and bigger numbers. (But it is fine for some sets, like $[1,5]$, whose maximal element is 5.) So we have to refine our question by being a bit more precise.

Definition 1.25. A set of numbers S is *bounded above* if there exists a number b such that for all $x\in S$, we have $x\leq b$. In this case, b is called an *upper bound* of S.

We can similarly define a *lower bound* of S (see the exercises).

See Figure 1.1 for a visual depiction of a set S of real numbers (on the number line) and some examples of upper and lower bounds of S.

[4]These axioms are also satisfied by lots of other fields, as you'll see in Algebra classes.

Example 1.26. The set $\{1, 2, 3\}$ is bounded above; some upper bounds are 3 (which is in fact its maximal element) and π and 400.

The set $\mathbb{R}^- = \{x \in \mathbb{R} \mid x < 0\}$ is bounded above (by any nonnegative number).

OK, so our first question was naive, but we can improve it to ask what we really meant.

1.3.2 Does every nonempty set which is bounded above contain a maximal element?

Well, again, the answer is NO. For example, consider the simple open interval $S = (1, 5)$. We'd like to say that 5 is the maximal element, but $5 \notin S$. And $4.9 \in S$ but it's not a maximal element since $4.95 \in S$ and $4.95 > 4.9$.

In fact, suppose to the contrary that $m \in S$ was a maximal element of S. So obviously $1 < m < 5$ since $m \in S$. But then

$$m = 2m/2 < (m + 5)/2 < (5 + 5)/2 = 5$$

so $(m + 5)/2 \in S$ and it is bigger than m. So m is not a maximal element. *There is no maximal element of* $S = (1, 5)$ *!*

That is quite frustrating; obviously 5 is the number we want to talk about, but we don't have a word for it.

What can we say about 5? Well, it is an upper bound of S, and in fact it is the smallest (least) upper bound of S. Tah-dah!

1.3.3 What do we mean by "the least upper bound of S"?

So the notion we are trying to nail down and define precisely is: s is the least upper bound of S iff

(a) s is an upper bound of S, and

(b) if $t < s$, then t is not an upper bound of S.

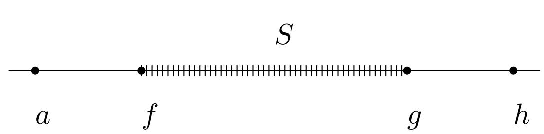


Figure 1.1: A set S of real numbers (with elements as indicated by little vertical dashes). The numbers a and f are lower bounds of S and the numbers g and h are upper bounds of S.

(Note that we are **not** saying that $s \in S$ or that $s \notin S$ — this definition covers both kinds of cases.)

In mathematical language ("useful language") the first condition (a) is equivalent to

$$\forall x \in S, \quad x \leq s.$$

For (b): how do we say "t is not an upper bound of S"? Well, recall that the logical negation of "$\forall x \in S, t \geq x$" is

$$\exists x \in S \ s.t. \ t < x.$$

Therefore condition (b) is

$$\forall t < s, \quad \exists x \in S \ s.t. \ t < x.$$

Perfect!

1.3.4 The definition of the supremum of a set

Definition 1.27. Let $S \subseteq \mathbb{R}$ be a nonempty set which is bounded above. Then $s \in \mathbb{R}$ is the *least upper bound of* S or the *supremum of* S, and we write $s = \sup(S)$, if:

(a) $\forall x \in S$, $x \leq s$, and

(b) $\forall t < s$, $\exists x \in S$ such that $t < x$.

In Figure 1.1, the supremum of S is g.

Example 1.28. The supremum of $S = \{1, 2, 3\}$ is 3, since it's an upper bound, and if $c < 3$, then c is not an upper bound since $x = 3$ is an element of S which is strictly greater than c. Therefore $\sup(S) = 3$.

In the previous example, $\sup(S)$ was an element of S, in fact the maximal element of S. This always happens when the set S has a maximal element, as follows.

Example 1.29. Suppose S has a maximal element m. Then by definition $\forall x \in S$, we have $x \leq m$. Moreover, if $t < m$, then $m \in S$ is an element of the set which satisfies $t < m$, so the second condition also holds. Thus $\sup(S) = m$.

Remark 1.30. The point of the previous example: if your set contains a maximum element, then that's the supremum, yes. What makes life interesting are all the cases when $\sup(S) \notin S$.

Example 1.31. Consider the set $S = \{\frac{1}{n} \mid n \in \mathbb{N}_+\}$. Note that it is bounded above by 1, which is in the set, hence is its supremum.

Example 1.32. If $S = (1, 5)$ then $\sup(S) = 5$. Why? We have shown that 5 is an upper bound of S. Let us show that no number $s < 5$ can be an upper bound. If $1 < s < 5$, then we showed this result earlier. If $s \leq 1$, then $s < 3 \in S$, so of course it is not an upper bound of S. Hence 5 is indeed the *least* upper bound of S.

Recall that our notation for intervals is as follows. Suppose $a < b$ then we have

$$\begin{aligned} (a,b) &= \{x \in \mathbb{R} \mid a < x < b\} && \text{an } \textit{open interval} \\ (a,b] &= \{x \in \mathbb{R} \mid a < x \leq b\} && \text{a } \textit{half-open interval} \\ [a,b) &= \{x \in \mathbb{R} \mid a \leq x < b\} && \text{a } \textit{half-open interval} \\ [a,b] &= \{x \in \mathbb{R} \mid a \leq x \leq b\} && \text{a } \textit{closed interval}. \end{aligned}$$

The supremum of any of these sets is b (exercise).

Remark 1.33. We also write $(-\infty, b)$ and (a, ∞) to denote unbounded intervals; this notation is consistent with the above. Note that we always use an open parenthesis on $\pm\infty$, since it is not logical to include $\pm\infty$, which is not a number, in the set.

Example 1.34. Let S be the empty set. Then every number is an upper bound (because the condition we need to check can't be false, so must be true). So the most consistent thing we can write is $\sup(S) = -\infty$.

Example 1.35. Let $S = \{1 - \frac{1}{n^2} \mid n \in \mathbb{N}_+\} = \{0, \frac{3}{4}, \frac{8}{9}, \cdots\}$. Let's verify that $\sup(S) = 1$.

We have to show that the two conditions hold.

(a) For all $n \in \mathbb{N}_+$, we have $\frac{1}{n^2} > 0$, so $1 - \frac{1}{n^2} < 1$. Thus every $x \in S$ satisfies $x < 1$, so 1 is an upper bound of S.

(2) Let $t < 1$ be a real number. *(My rough work: I want to find an n such that $1 - \frac{1}{n^2} > t$. I solve for n: $n^2 > \frac{1}{1-t}$ or $n > \sqrt{\frac{1}{1-t}}$. Got it!)* Since $t < 1$, $\frac{1}{1-t} > 0$. Choose any $n \in \mathbb{N}$ such that $n > \sqrt{\frac{1}{1-t}} > 0$. It follows that $n^2 > \frac{1}{1-t}$, whence $1 - t > \frac{1}{n^2}$, or $t < 1 - \frac{1}{n^2} \in S$, which is what we wanted to show.

We conclude that $s = 1$ is the supremum of S (or the least upper bound of S).□

Example 1.36. Show that if $E = \{x \in (0,1) \mid x \in \mathbb{Q}\}$ then $\sup(E) = 1$.

Solution: (a) If $x \in E$, then $0 < x < 1$, so it's true that 1 is an upper bound of E.

(b) Let $t < 1$. We have to show that there is an $x \in E$ such that $t < x$. Well, if $t < 0$ this is easy; we could just take $x = \frac{1}{2} \in E$. Otherwise, the decimal expansion of t has the form $0.a_1a_2a_3\cdots$ (where each a_i represents one decimal digit) since $t \in [0,1)$. Since $0.99999\cdots = 1$, we know that not every a_i is 9. Therefore there exists at least one digit, say the nth one a_n, which is not 9; set $b = a_n + 1$. Now define $x = 0.a_1a_2\cdots a_{n-1}b$. Then $x \in \mathbb{Q}$, $x \in (0,1)$ and $x > t$; so x satisfies all requirements.

Therefore, yes, $\sup(E) = 1$. □

Example 1.37. Prove that if $E = (-10, -5) \cup (-4, -3)$ then $\sup(E) = -3$.

Solution: (a) If $x \in E$, then either $-10 < x < -5 < -3$ or $-4 < x < -3$. In either case, $x < -3$. Thus -3 is an upper bound of E.

(b) Let $t < -3$. If $t < -4$, we can take $x = -3.5$, because then $x \in E$ and $x > t$. Otherwise, set $x = \frac{1}{2}(t - 3)$, which is the average of t and -3. Thus $-4 \leq t < x < -3$, whence $x \in E$ and $x > t$, as was required.

Thus $\sup(E) = -3$. $\square$

Note that in this example, we didn't need to take into account any elements of E which were far from the supremum; the whole proof was done with the elements of E that were close to $\sup(E)$. That ought to make sense.

Lemma 1.38. *Let S be a nonempty set and $s \in \mathbb{R}$. Then the statement*

$$\forall t < s \quad \exists x \in S \text{ s.t. } t < x$$

is equivalent to the statement

$$\forall \varepsilon > 0 \quad \exists x \in S \text{ s.t. } s - \varepsilon < x.$$

Proof. We have two proofs to make, one for each direction of the equivalence.

$\Rightarrow$: Suppose that it is true that $\forall t < s \quad \exists x \in S$ *s.t.* $t < x$. We need to show that for every $\varepsilon > 0$, there exists an $x \in S$ such that $s - \varepsilon < x$. So let $\varepsilon > 0$ and set $t = s - \varepsilon$; then $t < s$ so by our hypothesis, $\exists x \in S$ such that $x > t = s - \varepsilon$, which is what we needed to show.

$\Leftarrow$: Suppose now that $\forall \varepsilon > 0 \quad \exists x \in S$ *s.t.* $s - \varepsilon < x$. We need to show that for all $t < s$, there is an $x \in S$ such that $t < x$. So let t be a real number such that $t < s$. Then $s - t > 0$. Thus we may set $\varepsilon = s - t$ and $\varepsilon > 0$. Thus by our hypothesis, $\exists x \in S$ such that $x > s - \varepsilon = s - (s - t) = t$, which is what we needed to show. $\square$

Example 1.39. We saw that $\mathbb{R}^-$ is bounded above. We claim $\sup(\mathbb{R}^-) = 0$. There are two things to verify. First, yes it is an upper bound, since $0 \geq x$ for all $x \in \mathbb{R}^-$. Second, let $\varepsilon > 0$. Set $x = -\varepsilon/2$; then $0 - \varepsilon < -\varepsilon/2 \in \mathbb{R}^-$. Hence (1) and (2) are satisfied and so $\sup(\mathbb{R}^-) = 0$. Notice that $\sup(S) \notin S$ in this case. $\square$

Example 1.40. Prove, using the ε-version of the definition of the supremum, that if $E = \{\frac{3n+1}{n+1} \mid n \in \mathbb{N}\}$ then $\sup(E) = 3$.

Solution: Before starting, let's simplify the expression representing the elements of the set E. We have

$$\frac{3n+1}{n+1} = \frac{3(n+1) - 3 + 1}{n+1} = 3 - \frac{2}{n+1}$$

so now it's easy to see that 3 should be the supremum. But we need to prove it.

Proof: (a) If $x \in E$, then there is some $n \in \mathbb{N}$ such that $x = 3 - \frac{2}{n+1}$. Since $n \geq 0$, $2/(n+1) > 0$, whence $-2/(n+1) < 0$ and thus $3 - 2/(n+1) < 3$.

(b) Let $\varepsilon > 0$. After a bit of calculation on some scrap paper, we realize that what we need to do is choose a natural number n which is bigger than $2/\varepsilon - 1$ (which is possible, because $\mathbb{N}$ is not bounded above!). This would imply that $n + 1 > 2/\varepsilon$, which gives (since $\varepsilon > 0$ and $(n+1) > 0$) that $\varepsilon > 2/(n+1)$. Thus $-\varepsilon < -2/(n+1)$, which means $3 - \varepsilon < 3 - 2/(n+1)$. The number $x = 3 - 2/(n+1)$ is thus an element of E satisfying $3 - \varepsilon < x$, as was required to show.

Thus $\sup(E) = 3$. $\square$

And now: for an example that explains why we have done all this:

Example 1.41. Let $S = \{x \in \mathbb{Q} \mid x^2 < 2\}$. Then (as we will prove next) $\sup(S) = \sqrt{2}$ but (as we have seen) $\sqrt{2} \notin \mathbb{Q}$. So this is a subset of $\mathbb{Q}$ which does not admit a supremum in $\mathbb{Q}$. That is: $\mathbb{Q}$ has holes!

Proof: (a) By definition, $x \in S$ implies $x < \sqrt{2}$, so $\sqrt{2}$ is indeed an upper bound of S.

(b) Let $\varepsilon > 0$. If $\varepsilon > 1$ choose $x = 1$; then $x \in S$ and $\sqrt{2} - \varepsilon < \sqrt{2} - 1 < x$ and we are done. Otherwise, let $0.a_1a_2a_3\cdots$ be the decimal expansion of ε, where each a_i represents a digit. Since $\varepsilon > 0$, not all digits are 0; let a_n be the first nonzero digit, in the nth place after the decimal point; then $\varepsilon \geq a_n * 10^{-n} \geq 10^{-n}$. Let $1.41421356237\cdots$ be the decimal expansion of $\sqrt{2}$. Let x be the rational number given by taking the first n decimal digits of $\sqrt{2}$: so if $n = 4$ we take $x = 1.4142$. Then $x < \sqrt{2}$ and $x \in \mathbb{Q}$ so $x \in S$. Even better, by our construction, $\sqrt{2} - x < 10^{-n} < \varepsilon$, whence $\sqrt{2} - \varepsilon < x$, as was required to show.

Thus $\sqrt{2} = \sup(S)$.

1.3.5 Exercises

1. Define the infimum of a set, also known as the greatest lower bound, in analogy with the definition of the supremum. We write $\inf(S)$.
2. Let S be a set which is bounded above. Show that $b = \sup(S)$ if and only if the following two conditions hold:
   (a) b is an upper bound of S, and
   (b) for any $c < b$ there is some $x \in S$ such that $c < x \leq b$.
3. Write down a theorem about $\inf(S)$, analogous to Lemma 1.38. Prove your theorem.
4. Prove that if S is a bounded set and $-S = \{-s \mid s \in S\}$ is the set obtained by replacing every element with its additive inverse (or, multiplying every element of the set by -1) then $-\sup(S) = \inf(-S)$ and $-\inf(S) = \sup(-S)$.
5. Find the supremum and infimum of each of the following sets in $\mathbb{R}$, if they exist. State in each case if $\sup(S) \in S$ and if $\inf(S) \in S$.
   (a) Fix $n \in \mathbb{N}$. $S = \{\frac{p}{q} \mid p, q \in \mathbb{N}, q \neq 0, p + q = n\}$
   (b) $S = \{x \mid x^2 < 16\}$
   (c) $S = \{x \mid x^2 < -1\}$
   (d) $S = \{x \mid (x^2 + 1)^{-1} > \frac{1}{2}\}$
   (e) $S = \{x \mid x \in \mathbb{Z}, x^2 < 7\}$
   (f) $S = \{x \mid x \in \mathbb{Q}, x^2 < 7\}$
   (g) $S = \{x \mid |2x + 3| < 7\}$
6. Prove the following for any bounded sets S, T of real numbers:
   (a) if $S + T = \{s + t \mid s \in S, t \in T\}$ then $\sup(S + T) = \sup(S) + \sup(T)$.
   (b) if $S \cup T = \{x \mid x \in S \text{ or } x \in T\}$ (the union of S and T) then $\sup(S \cup T) = \max\{\sup(S), \sup(T)\}$
   (c) if $S \cap T = \{x \mid x \in S \text{ and } x \in T\}$ (the intersection of S and T) then $\sup(S \cap T) \leq \min\{\sup(S), \sup(T)\}$. Give an example to show that equality need not hold.

1.4 The real numbers

1.4.1 The completeness axiom

The Completeness Axiom: An ordered field F satisfies the completeness axiom if for every nonempty set $S \subseteq F$ which is bounded above, $\sup(S) \in F$.

Example 1.42. Consider the set $S = \{x \in \mathbb{Q} \mid x^2 < 2\}$. Then $S \subseteq \mathbb{Q}$, but we have shown that $\sup(S) = \sqrt{2} \notin \mathbb{Q}$. So $\mathbb{Q}$ doesn't satisfy the completeness axiom!

Theorem 1.43 (Existence of the real numbers)**.** *The set of real numbers $\mathbb{R}$ is the unique ordered field*[5] *which satisfies the completeness axiom.*

We accept this theorem as true for this course; one proves it in higher level courses. The uniqueness comes from showing that if you are an ordered field, you must contain $\mathbb{Q}$, and then the following ideas do the rest:

Sketch of the proof of the completeness of $\mathbb{R}$. In terms of our working definition: we know how to proceed to get the supremum. Suppose S is a nonempty subset of $\mathbb{R}$ which is bounded above. Choose $x \in S$ and b and upper bound of S; suppose $x \geq 0$. (If $x < 0$, replace the set S by $S' = \{|x| + s \mid s \in \S\}$ so that the supremum will be positive; then prove that $\sup(S') = |x| + \sup(S)$ (exercise) so that you recover the result for S.)

Create the decimal expansion $a_0.a_1a_2\cdots$ for $\sup(S)$ by choosing each decimal to be the largest digit a_n such that $a_0.a_1a_2\cdots a_n$ is not an upper bound of S but $a_0.a_1a_2\cdots a_n + 10^{-n}$ is an upper bound of S.

By construction, the infinite decimal number a thus constructed has the property that for every $\varepsilon > 0$, $a + \varepsilon$ is an upper bound of S, and $a - \varepsilon$ is not an upper bound for S (exercise). Thus we may deduce that $a = \sup(S)$ (exercise).

We can't do this in the rational numbers because we can't be sure that this process of constructing a terminates in finite time (or has a simple pattern). □

Conversely, note that every real number occurs as a supremum of a set of rational numbers, so in fact we did need all the real numbers to complete $\mathbb{Q}$:

Example 1.44. Let x be a real number with decimal expansion $x = a.a_1a_2a_3\cdots$ for an integer a and digits a_i. Then $x = \sup\{a, a.a_1, a.a_1a_2, \cdots\}$; that is, each decimal number is the supremum of at least one set of rational numbers, hence by our definition, is a real number.

This definition can be useful, too.

Example 1.45. Let $S = \{x \in \mathbb{Q} \mid x^2 < 2\}$. Then we can define $\sqrt{2} = \sup(S)$ (and we can show that this is exactly what we wanted $\sqrt{2}$ to mean).

[5] up to renaming the symbols for addition, multiplication and the order, for example

Example 1.46. Let $x \in \mathbb{R}$ and set $S = \{n \in \mathbb{N} \mid n \leq x\}$. Then S is nonempty and bounded above (by x) so has a supremum. This is an integer called the floor of x, and denoted $\lfloor x \rfloor$. Note that $x - 1 < \lfloor x \rfloor$, since otherwise, we would have $x - 1 \geq \lfloor x \rfloor$ or $x \geq \lfloor x \rfloor + 1$, contradicting the maximality of $\lfloor x \rfloor$.

1.4.2 Two properties of $\mathbb{Z}$

Example 1.47. Let $S = \mathbb{N}$ be the set of natural numbers. Then S does not have an upper bound. We write $\sup(S) = \infty$.

The archimedean property: $\mathbb{N}$ is not bounded above.

This is called the *archimedean property*[6], of $\mathbb{R}$. It is equivalent to the following, more useful, properties:

- For every $\varepsilon > 0$ there exists some $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \varepsilon$.
- For every $x \in \mathbb{R}^+$ and for every $b \in \mathbb{R}$ there is some $n \in \mathbb{N}$ such that $nx > b$.

Why are these statements all equivalent? Notice tht $1/n < \varepsilon$ is equivalent to $n > \varepsilon^{-1}$, so it's the statement that ε^{-1} isn't an upper bound on $\mathbb{N}$. Similarly, $nx > b$ is saying $n > bx^{-1}$, that is, bx^{-1} isn't an upper bound on $\mathbb{N}$.

The well-ordering principle: Every non-empty subset of $\mathbb{N}$ has a least element.

This is fairly obvious, since $\mathbb{N} = \{0, 1, 2, 3 \cdots\}$ — and yet is quite profound. It is a lot like the completeness property, in that it promises you can't find a set of natural numbers so weird that its lower bound is a non-integer.

Compare this to $\mathbb{R}$, where the well-ordering principle is FALSE: for example $(1, 5)$ is a nonempty subset of $\mathbb{R}$ which does not contain a least element. This is the kind of problem that led us to define the supremum (more exactly, the infimum) of a set.

1.4.3 The density of $\mathbb{Q}$ in $\mathbb{R}$

The archimedean property allows us to prove a really amazing result:

Theorem 1.48. *Between every two distinct real numbers there is a rational number and also an irrational number.*

[6]This was named for Archimedes because it appeared as an axiom in his book "On the Sphere and Cylinder" (225 BC) although Archimedes credits its discovery to Eudoxus of Cnidus, who lived over a century earlier (around 400 BC); this property was also mentioned in Euclid's Elements (around 300 BC). The name "archimedean property" was only coined as a consequence of mathematicians conceiving of number systems that did not have this property, and hence are now called "nonarchimedean." Source: Wiki and the excellent on-line *The MacTutor History of Mathematics archive.*

Heuristic proof (using our working definition of $\mathbb{R}$): Since $x \neq y$, their decimal expansions are not identical. Identify the point at which they differ, and choose a rational number with finite decimal expansion lying between them. For an irrational number: complete your decimal with any non-repeating pattern (like $121121112\cdots$).

More solid proof. So let $x, y \in \mathbb{R}$, $x < y$. Using the archimedean property, we can find $q \in \mathbb{N}$ such that

$$q > \frac{1}{y-x} \quad \text{or } 1 < q(y-x).$$

Choose the smallest integer p such that $p > qx$; then $p - 1 < qx$ or $p < qx + 1$. So

$$qx < p < qx + 1 < qx + q(y-x) = qy$$

and after dividing by q, we get

$$x < \frac{p}{q} < y.$$

Success!

For an irrational number: Find a rational number r between x and y and then another rational number s between r and y; then

$$x < r < s < y$$

and the number $r + \frac{1}{\sqrt{2}}(s-r)$ is an irrational number lying between r and s (details as exercise). □

An immediate consequence is that between every two numbers there are infinitely many rational and irrational numbers. (exercise)

1.4.4 Mathematical Induction

Mathematical induction is a particular kind of argument that applies to statements which hold "for every $n \in \mathbb{N}$". It was developed in 1575, through the work of Maurolycus [B], although its discovery is typically attributed to Blaise Pascal (1623-1662). It is based on the well-ordering principle.

The kinds of statements we can use induction to prove are the following:

- $P(n)$: For every $n \in \mathbb{N}$, $\displaystyle\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$
- $P(n)$: For every integer $n \geq 1$ and any $r \in \mathbb{R}$, $r \neq 1$, we have $\displaystyle\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$
- $P(n)$: For every $n \in \mathbb{N}$, $\displaystyle\sum_{k=0}^{n} k^3 = \left(\sum_{k=0}^{n} k\right)^2$
- $P(n)$: For every $n \geq 4$, $n \in \mathbb{N}$, $n^2 \leq 2^n$

Mathematical induction works as follows:

1. (base case) Prove a base case (like $n = 0$ or $n = 1$), i.e. prove the statement $P(0)$ (or $P(1)$, or whatever the first one should be).
2. (inductive hypothesis) Assume that the statement is true for a certain value of n, which should be $\geq$ your base case. That is, assume $P(n)$ is true.
3. (induction step) Prove that the statement is true for the next value of n, that is, prove that $P(n+1)$ is true.

The idea is this: Say your base case is $n = 1$. Then by setting $n = 1$ in your inductive hypothesis, your induction step ensures that case $n + 1 = 2$ is true (since your inductive hypothesis is a fact in that case). And then setting $n = 2$ instead, you get that $n + 1 = 3$ must be true. And so on. In other words, by letting n be a variable we are doing infinitely many proofs at once, and altogether they ensure that we proved for all n.

Remark 1.49. Induction relies on the *well-ordering principle* of $\mathbb{N}$, which promises that a nonempty subset of $\mathbb{N}$ has an infimum which lies in $\mathbb{N}$.

Here, in this context, this means: if your statement is false, then the set of counterexamples (which is a subset of $\mathbb{N}$) is nonempty, so it has a least element. Let that least element be N. A proof by induction guarantees that if the case $N - 1$ is true, then the case N is true. But the case N is false, so the case $N - 1$ is false. But N was the least counterexample! Contradiction. Hence the set of counterexamples is false.

We can now appreciate the import of the well-ordering principle: it is FALSE for real numbers: some sets have no lower bounds, and others have an infimum but the infimum is NOT IN THE SET. The Well-Ordering Principle promises that there is an infimum but that it actually lies in the set.

1.4.5 Exercises

1. Prove that between any two real numbers there are infinitely many rational numbers and infinitely many irrational numbers.
2. Give an example of a nonempty subset of $\mathbb{Q}$ which is bounded above but whose supremum does not lie in $\mathbb{Q}$. Conclude that $\mathbb{Q}$ is not complete.
3. Prove using the Archimedean principle that for every $\varepsilon > 0$ there exists some $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \varepsilon$.
4. Prove using the Archimedean principle that for every $x \in \mathbb{R}^+$ and for every $b \in \mathbb{R}$ there is some $n \in \mathbb{N}$ such that $nx > b$.
5. Show that every nonempty subset of $\mathbb{Z}$ which is bounded above has its supremum in $\mathbb{Z}$. Show, however, that $\mathbb{Z}$ is not a *field*, that is, it fails one of the axioms A1-4, M1-4, D1 (which one(s)?).
6. Show that a nonempty subset of the integers which is bounded above will have a supremum, and that supremum will be an integer. Conclude that for any $S \subset \mathbb{Z}$ which is nonempty and bounded above, $\sup(S) \in \mathbb{Z}$.
7. Do the same with subsets of integers bounded below, to conclude the well-ordering principle.

1.5 Applications of mathematical induction

Lemma 1.50. *Show that for all $n \in \mathbb{N}$, $\sum_{k=0}^{n} k^3 = \left(\sum_{k=0}^{n} k\right)^2$.*

Recall

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}. \tag{1.2}$$

Proof. (1) The base case here is $n = 0$. Both expressions equal 0 in this case, so are equal.

(2) Now suppose there is some $n \geq 0$ for which

$$\sum_{k=0}^{n} k^3 = \left(\sum_{k=0}^{n} k\right)^2. \tag{1.3}$$

Using this induction hypothesis, we must show that

$$\sum_{k=0}^{n+1} k^3 = \left(\sum_{k=0}^{n+1} k\right)^2.$$

We scribble a bit on some scrap paper, and eventually discover the following chain of equalities, such that each equality is obvious and the whole constitute a proof:

$$\begin{aligned}
\sum_{k=0}^{n+1} k^3 &= \sum_{k=0}^{n} k^3 + (n+1)^3 \\
&= \left(\sum_{k=0}^{n} k\right)^2 + (n+1)^3 \quad \text{by our induction hypothesis (1.3)} \\
&= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \quad \text{by (1.2)} \\
&= \frac{1}{4}(n+1)^2 \left(n^2 + 4(n+1)\right) \quad \text{factoring} \\
&= \frac{1}{4}(n+1)^2(n+2)^2 \\
&= \left(\frac{(n+1)(n+2)}{2}\right)^2 \\
&= \left(\sum_{k=0}^{n+1} k\right)^2 \quad \text{by the relation (1.2).}
\end{aligned}$$

Thus the inductive step is true, and so by the principles of mathematical induction, the equality is true for all $n \geq 0$. □

Lemma 1.51. *Show that for all $n \geq 4$, $n \in \mathbb{N}$, $n^2 \leq 2^n$.*

Proof. (1) Here, the based case is $n = 4$. When $n = 4$, $n^2 = 16$ and $2^n = 16$, so $n^2 \leq 2^n$, and it's true.

(2) Now suppose there is some $n \geq 4$ for which $n^2 \leq 2^n$. Using this hypothesis, we must show that

$$(n + 1)^2 \leq 2^{n+1}.$$

Good; after scribbling on some paper we find the following chain of inequalities (for example — there are other solutions possible!):

$$\begin{aligned}
(n+1)^2 &= n^2 + 2n + 1 \\
&\leq n^2 + 4n \quad \text{since } 2n > 1 \\
&\leq n^2 + n^2 \quad \text{since } n > 4 \\
&\leq 2^n + 2^n \quad \text{by the inductive hypothesis} \\
&= 2^{n+1}.
\end{aligned}$$

Thus $n^2 \leq 2^n$ implies that $(n + 1)^2 \leq 2^{n+1}$, so by the principles of mathematical induction, the inequality holds for all $n \geq 4$. □

We never did prove anything about $n < 4$; induction only goes in one direction.

Lemma 1.52. *Prove that for any $r \neq -1$, and every integer $n \geq 1$, $\displaystyle\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}$.*

Proof. (Base case) When $n = 1$ the formula is $1 + r = \frac{1-r^2}{1-r}$, which is true.

(Induction hypothesis) Assume that the formula holds for $n = k$, where $k \geq 1$. That is, we assume

$$\sum_{i=0}^{k} r^i = \frac{1 - r^{k+1}}{1 - r}.$$

(Induction step) Let us prove the formula for $n = k + 1$. We note that

$$\begin{aligned}
\sum_{i=0}^{k+1} r^i &= 1 + r + \cdots + r^{k+1} \\
&= (1 + r + \cdots + r^k) + r^{k+1} \\
&= \frac{1 - r^{k+1}}{1 - r} + r^{k+1} \quad \text{by inductive hypothesis} \\
&= \frac{1 - r^{k+1}}{1 - r} + \frac{r^{k+1} - r^{k+2}}{1 - r} \\
&= \frac{1 - r^{k+2}}{1 - r}
\end{aligned}$$

as required. Therefore the formula holds with $n = k + 1$, and by the principle of mathematical induction, it holds for all $n \geq 1$. □

Lemma 1.53. *For every $n \geq 1$, for every $x > -1$, $(1 + x)^n \geq 1 + nx$.*

Proof. (Base case) Let $n = 1$ and let $x > -1$. Then $(1+x)^n = (1+x)^1 = 1+x$ and $1+nx = 1+x$ so the inequality holds.

(Inductive hypothesis) Suppose that for $n = k$, the inequality holds for all $x > -1$, that this, for all $x > -1$ we have

$$(1+x)^k \geq 1 + kx.$$

(Induction step) Now set $n = k+1$. We have

$$\begin{aligned}
(1+x)^{k+1} &= (1+x)(1+x)^k \\
&\geq (1+x)(1+kx) \quad \text{since } (1+x) > 0\text{, and by the inductive hypothesis} \\
&= 1 + x + kx + kx^2 \\
&\geq 1 + (k+1)x \quad \text{since } kx^2 > 0.
\end{aligned}$$

Therefore the result holds for $n = k+1$. By the principle of mathematical induction, the inequality holds for all $n \geq 1$. ☐

Corollary 1.54. *For any $y > 1$ and $b \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $y^n > b$ for all $n \geq N$. Also, for any $0 < y < 1$ and $b > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $y^n < b$.*

Proof. First suppose $y > 1$. Then set $x = y - 1 > 0$ so that $y = 1 + x$. Choose, by the archimedean property, $N \in \mathbb{N}$ such that $Nx > b$. Then for any $n \geq N$, we have $nx > b$. Then by the previous theorem we have

$$y^n = (1+x)^n \geq 1 + nx > nx > b$$

as required.

Now suppose $0 < y < 1$. Then $y^{-1} > 1$ so by the first part, there is some N such that for all $n \geq N$, $(y^{-1})^n > b^{-1}$. Since all values are positive, multiplying through by $y^n b$ yields $y^n < b$, as required. ☐

Remark 1.55. We now have two handy results:

- For any $\varepsilon > 0$ there is some $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \varepsilon$ (by the archimedean property), and
- For any $\varepsilon > 0$ and $0 < x < 1$ there is some $n \in \mathbb{N}$ such that $x^n < \varepsilon$.

Both of these express the idea that the numbers $1/n$ and x^n, although never 0 and always positive, tend towards 0, and are so are great examples of *convergent sequences*.

1.5.1 Exercises

1. Express each of the following sums using summation notation:
   (a) $1 + 2 + 3 + \cdots + n$
   (b) $2 + 4 + 6 + \cdots + 20$
   (c) $f(-3) + f(-2) + f(-1) + f(0) + \cdots + f(14) + f(16)$

2. Write out each of the following in long form (using $\cdots$ as necessary):

   (a) $\sum_{i=0}^{n} i^3$

   (b) $\sum_{i=1}^{k} ki$ (Remember: the index i is the variable, whereas the k must be a constant here!)

   (c) $\sum_{i=1}^{k} x^i$

   (d) $f(-3) + f(-2) + f(-1) + f(0) + \cdots + f(14) + f(16)$

3. Prove the following propositions by induction.

   (a) For every $n \in \mathbb{N}$, $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

   (b) For every $n \in \mathbb{N}$, $\sum_{k=0}^{n} k^3 == \left(\sum_{k=0}^{n} k\right)^2$

   (c) For every $n \geq 4$, $n \in \mathbb{N}$, $n^2 \leq 2^n$ (so here your base case is $n = 4$, not $n = 0$ or $n = 1$)

   (d) For any $n \in \mathbb{N}$, if $a_1, \cdots, a_n$ are arbitrary real numbers, then

   $$|a_1 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$$

   (so here your induction is on n, the number of terms you are adding together).

   (e) For any $n \in \mathbb{N}$, if $a_1, \cdots, a_n$ are nonnegative real numbers then

   $$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geq 1 + a_1 + a_2 + \cdots + a_n$$

   (f) $\sum_{i=0}^{n} (-1)^i i^2 = \frac{(-1)^n n(n+1)}{2}$

   (g) $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$

   (h) $\sum_{i=1}^{n} (2i - 1) = n^2$

   (i) $n! \geq 2^n$ for all $n \geq 4$

4. Consider the statement

   $$1 + 2 + \cdots + n = \frac{(n+2)(n-1)}{2}$$

   Show that if the statement holds for $n = k$ then it must also be true for $n = k + 1$. Next: is there any $n \in \mathbb{N}$ for which this statement is true? (Is this a counterexample to the principle of mathematical induction? Discuss.)

5. The *factorial* of $n \in \mathbb{N}$ is defined as

   $$0! = 1, \quad \text{and for } n \geq 1, \quad n! = 1 \cdot 2 \cdot 3 \cdots n$$

and for any $0 \leq k \leq n$, we define the binomial coefficient (which we read as "*n choose k*") to be

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

(a) Prove that the binomial coefficients satisfy the relation

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

(You can prove this directly, without induction.)

(b) Use induction and part (a) to prove that the binomial coefficients are positive integers (for every n and k). (Hint: do induction on n and prove that for all $0 \leq k \leq n$ that $\binom{n}{k}$ is a positive integer. Oh: and don't re-use k for your inductive hypothesis, pick another letter for your variable instead.)

(c) Now prove the *binomial theorem* by induction:

Binomial Theorem: for any $n \in \mathbb{N}$ and for all $a, b \in \mathbb{R}$ such that $ab \neq 0$, we have

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

(This is a very useful theorem! Note that the binomial coefficients are thus just the entries of Pascal's triangle.)

6. Let f_n be the Fibonacci sequence: $f_1 = 1$, $f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. Prove the following relations by induction:

(a) $\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$

(b) $f_n \geq (3/2)^{n-2}$

(c) $\sum_{i=1}^{n} f_i = f_{n+2} - 1$

7. Let a_n be the sequence defined by $a_1 = 1$, $a_2 = 8$ and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$. Prove by induction that $a_n = 3(2^{n-1} + 2(-1)^n$ for all $n \geq 1$.

Chapter 2

Sequences

We now have the tools we need to talk about one of the main ideas in analysis: sequences of real numbers.

2.1 Sequences: definitions and examples

Definition 2.1. A *sequence* of real numbers is a function $a\colon \mathbb{N} \to \mathbb{R}$.

Some remarks:

- We write $a_0, a_1, a_2, a_3, \cdots$ instead of $a(0), a(1), a(2), a(3), \cdots$.
- Sometimes we will start our sequences at 1 instead of 0 (or even higher).
- We often denote the sequence just by giving the range (if the function is obvious enough): for example $\{a_n \mid n \in \mathbb{N}\}$ or $\{a_n\}_{n\geq 1}$ or even just $\{a_n\}$.

Some examples:

- $a_n = \frac{1}{n}$, $n \geq 1$ or $\{1/n\}_{n\geq 1}$ or $\{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$
- $b_n = 1 + \frac{1}{2}n$, $n \geq 0$ or $\{1 + \frac{1}{2}n \mid n \geq 0\}$
- $c_n = \frac{n}{n+1}$, $n \geq 0$
- $d_n = \frac{(-1)^n}{n^2}$, $n \geq 1$
- $e_n = (-1)^n$, $n \geq 0$
- $g_n = 2^{-n}$, $n \geq 0$

Sequences can also be defined recursively, like:

- $f_1 = 1,\ f_2 = 1,\ f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$ (this is the Fibonacci sequence $1, 1, 2, 3, 5, 8, \cdots$)
- $h_1 = 1,\ h_{n+1} = \frac{h_n^2+2}{2h_n}$ for $n \geq 2$

A very important example (to us) is the following:

$p_n =$ first n digits of π: $\{3, 3.1, 3.14, 3.141, 3.1415, \cdots\}$

In fact, we can think of our working definition of real numbers as saying: every real number is a *sequence of rational numbers* (in the above way).

2.1.1 Exercises

1. Which of the following denote sequences? (We often use a variety of notation interchangeably; and we often are lazy and do not explicitly say that a variable is in $\mathbb{Z}$ (especially when the variable is named something like k, ℓ, m, n which are traditional letters for integer variables) — what is important about being a sequence is that it meets the definition.)
   (a) $\{1, 2, 3\}$
   (b) $\{1, 1, 1, 1, \cdots\}$
   (c) $\{\frac{1}{x} \mid x \in \mathbb{R}, x > 0\}$
   (d) $\{1, 3, 1, 2, 543, 2, 45, 23, 336, \cdots\}$
   (e) $a_k = 1$ if $1 \leq k \leq 17$, $a_k = \frac{1}{k}$ if $k > 17$
   (f) $a_k = (-1)^k k!/(k+1)$ for $k \geq 0$
   (g) $a_\ell =$ the temperature in Ottawa at noon on day ℓ, with $\ell = 0$ corresponding to January 1, 2012.
   (h) $a_1 = 9$, $a_n = a_{n-1} + 1$ for all $n \geq 2$.
   (i) $a_k^2 = k^2$ for all $k \geq 1$.
   (j) $a_m = \sin(m)$, for all $m \geq 0$.
2. Does every sequence have to be defined by a formula?
3. Does every sequence have to have a "limit"?
4. If a sequence $\{a_n\}_{n\geq 0}$ has a limit L, does that mean that for n sufficiently large, we have $a_n = L$?
5. Give an example of a sequence that is always increasing but does not go to infinity.

2.2 The limit of a sequence

2.2.1 Getting the right definition of "the limit"

There a many properties that we can ask of a sequence, and we'll describe many of them, but the most important concept is that of a *convergent* sequence, that is, a sequence that tends to a limit.

It took a long time (historically) to formulate this concept — one had to decide whether a sequence like e_n should count as convergent or not, for example — and what has been universally agreed upon is the following definition.

Definition 2.2. Let $\{a_n \mid n \geq 1\}$ be a sequence (of real numbers). A number a is called the *limit of the sequence* if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, |a - a_n| < \varepsilon.$$

We can also write this as:

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ such that } n \geq N \text{ implies } |a - a_n| < \varepsilon.$$

If a limit exists, then we say that the sequence is *convergent* and that it *converges to the limit a*. In this case we write

$$a = \lim_{n\to\infty} a_n.$$

If there is *no* number a that satisfies the above condition, then we say that the sequence *diverges* (or "does not converge").

Notice that the definition doesn't say how to *find* the limit; just how to recognize if the number you have is indeed the limit.

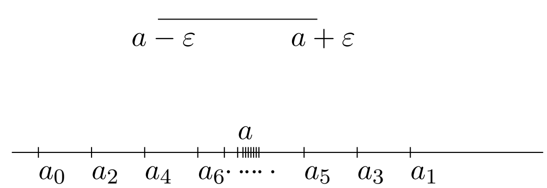


Figure 2.1: A sequence $\{a_n\}_{n\in\mathbb{N}}$ of real numbers (with elements as indicated by little vertical dashes). This sequence converges to the limit a: for every $\varepsilon > 0$, the interval $(a - \varepsilon, a + \varepsilon)$ of radius ε centered at a (sample drawn above the number line) does not necessarily contain the entire sequence, but does contain every element of the sequence from some point onwards.

2.2.2 Examples

Example 2.3. The sequence $\{p_n\}_{n\geq 0} = \{3, 3.1, 3.14, 3.141, \cdots\}$ was defined exactly so that its limit would be π, since by construction $0 < \pi - p_n < 10^{-n}$. Let's see how this proves that the limit is π. Let $\varepsilon > 0$ be arbitrary. By Corollary 1.54 there exists $N \in \mathbb{N}$ so that $10^{-N} < \varepsilon$; then for any $n \geq N$, we have

$$|\pi - p_n| < 10^{-n} \leq 10^{-N} < \varepsilon$$

as required.

Remark 2.4. So this example says further: we said that our "real number" was the sequence of successive decimal expansions; but actually, we want to identify our real number as the *limit of this sequence*. That's better; you could have come up with zillions of different sequences that have just as much right to be identified with π, but there's only one limit. And this is the definition that you use in MAT3120 to define the real numbers.

Example 2.5. Show that $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

Solution: We need to show that for every $\varepsilon > 0$, there is some integer N (depending on ε) such that for all $n \geq N$, the following inequality holds:

$$\left| \frac{n}{n+1} - 1 \right| < \varepsilon.$$

So let's manipulate this inequality until we understand for which n it will hold:

$$\left| \frac{n}{n+1} - 1 \right| = \left| \frac{n}{n+1} - \frac{n+1}{n+1} \right| = \frac{|-1|}{|n+1|} = \frac{1}{n+1}$$

and this is less than ε exactly when $n > \frac{1}{\varepsilon} - 1$. Good. So we can do the proof:

Proof: Let $\varepsilon > 0$ be arbitrary. Let $N \in \mathbb{N}$ be an integer which is greater than ε^{-1}, so that $\frac{1}{N+1} < \varepsilon$. Then for $n \geq N$, we have

$$\left| \frac{n}{n+1} - 1 \right| = \frac{1}{n+1} < \frac{1}{N+1} < \varepsilon$$

as required. We conclude that $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

Example 2.6. Find the limit of the sequence $\left\{ \dfrac{4n^2 - 3n + 2}{2n^2 + 2n - 1} \right\}_{n \in \mathbb{N}_+}$.

We start by simplifying this rational function of n. Since the degree of the numerator is (greater than or) equal to the degree of the denominator, we can use polynomial division:

$$\begin{array}{r@{}l} & \quad 2 \\ 2n^2 + 2n - 1 \; & \overline{)4n^2 - 3n + 2} \\ & -\, (\underline{4n^2 + 4n - 2}) \\ & \qquad\quad -\, 7n - 4 \end{array}$$

which gives

$$\frac{4n^2 - 3n + 2}{2n^2 + 2n - 1} = 2 - \frac{7n + 4}{2n^2 + 2n - 1}.$$

Thus, we feel confident the limit is 2. Let's prove it.

Rough work: We have

$$\left| \left(2 - \frac{7n+4}{2n^2 + 2n - 1} \right) - 2 \right| = \frac{7n+4}{2n^2 + 2n - 1}$$

We certainly don't want to try to solve $\frac{7n+4}{2n^2+2n-1} = \varepsilon$! But since we only want to arrive at the conclusion that this fraction is $< \varepsilon$, consider the following:

$$\frac{7n+4}{2n^2 + 2n - 1} \leq \frac{7n+4n}{2n^2 + 2n - 1} < \frac{11n}{2n^2} = \frac{11}{2n}.$$

The first inequality holds because $7n + 4 \leq 7n + 4n$ for all $n \geq 1$. For the second, we noted that $2n - 1 > 0$, so $2n^2 + 2n - 1 > 2n^2$. Thus the fraction on the right has a smaller denominator, so represents a larger number (since everything in sight is positive).

And with that, it is now easy to prove the limit is 2!

Let $\varepsilon > 0$. Choose, using the archimedian property, $N \in \mathbb{N}$ such that $N > \frac{11}{2\varepsilon}$. Then let $n \geq N$. We calculate

$$\begin{aligned}
\left| \left(\frac{4n^2 - 3n + 2}{2n^2 + 2n - 1} \right) - 2 \right| &= \left| \left(2 - \frac{7n + 4}{2n^2 + 2n - 1} \right) - 2 \right| \\
&= \frac{7n + 4}{2n^2 + 2n - 1} \\
&\leq \frac{7n + 4n}{2n^2 + 2n - 1} \quad \text{since } 4 \leq 4n \\
&< \frac{11n}{2n^2} \quad \text{since } 2n - 1 > 0 \\
&= \frac{11}{2n} \\
&\leq \frac{11}{2N} \quad \text{since } n \geq N \\
&\leq \frac{11}{2(11/(2\varepsilon))} \quad \text{since } N > 11/(2\varepsilon) \\
&= \varepsilon.
\end{aligned}$$

which is all we needed to show.

Example 2.7. Let $a_n = \sqrt{2 + \frac{1}{n}}$ for each $n \geq 1$. Show that $\lim_{n\to\infty} a_n = \sqrt{2}$.

We want $|\sqrt{2 + \frac{1}{n}} - \sqrt{2}| < \varepsilon$. We use the conjugate; so we want

$$\frac{(2 + \frac{1}{n}) - 2}{\sqrt{2 + \frac{1}{n}} + \sqrt{2}} < \varepsilon \tag{2.1}$$

We have to find an n for which this is true (and then expect that it will also be true for all greater n). Solving for n is extremely difficult! But wait a minute: since $2 + \frac{1}{n} \geq 2$ and $\sqrt{2} > 1$, we can say

$$\sqrt{2 + \frac{1}{n}} + \sqrt{2} > 1 + 1 = 2$$

(for example). So all we need to do is to find n such that the numerator is $< 2\varepsilon$! (Note: we are NOT saying that EVERY solution to (2.1) is obtained by choosing $\frac{1}{n} < 2\varepsilon$; all we are saying is that IF we choose n so that $\frac{1}{n} < 2\varepsilon$ THEN (2.1) holds.

Proof: Let $\varepsilon > 0$. By the Archimedian property, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < 2\varepsilon$. So for every $n \geq N$, we have

$$
\begin{aligned}
|\sqrt{2+\frac{1}{n}}-\sqrt{2}| &= \frac{(2+\frac{1}{n})-2}{\sqrt{2+\frac{1}{n}}+\sqrt{2}} \quad \text{by multiplying by } \frac{x}{x}, \text{ where } x \text{ is the conjugate} \\
&= \frac{\frac{1}{n}}{\sqrt{2+\frac{1}{n}}+\sqrt{2}} \\
&< \frac{\frac{1}{n}}{2} \quad \text{since } \sqrt{2+\frac{1}{n}}+\sqrt{2} > 2 \\
&< \frac{2\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

Example 2.8. Let $y > 0$. Show that $\lim_{n\to\infty} \sqrt[n]{y} = 1$. (Recall that for positive y, $z = \sqrt[n]{y}$ iff $z^n = y$ and $z > 0$.)

Solution: We need to solve the inequality

$$|\sqrt[n]{y} - 1| < \varepsilon$$

which by Lemma 1.20 is equivalent to

$$-\varepsilon < \sqrt[n]{y} - 1 < \varepsilon$$

or

$$1 - \varepsilon < \sqrt[n]{y} < 1 + \varepsilon.$$

We want to take nth powers of both sides, but this preserves inequalities only if all terms are positive (exercise), so we'd have a problem if $\varepsilon > 1$. So let's assume that $\varepsilon < 1$ for the moment:

$$(1-\varepsilon)^n < y < (1+\varepsilon)^n.$$

Now we apply Corollary 1.54.

Since $1 + \varepsilon > 1$, there is some $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $(1+\varepsilon)^n > y$.

Since $0 < 1 - \varepsilon < 1$, and $y > 0$, there is some $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, we have $(1-\varepsilon)^n < y$.

So we have all the ingredients we need; let's do the proof.

Proof. Suppose $y > 0$. Let $\varepsilon > 0$. Set $\varepsilon' = \min\{\frac{1}{2}, \varepsilon\}$, so that $\varepsilon' \leq \varepsilon$ and $\varepsilon' < 1$. Since $1 + \varepsilon' > 1$, there is some $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $(1+\varepsilon')^n > y$. Since $0 < 1 - \varepsilon' < 1$, and $y > 0$, there is some $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, we have $(1-\varepsilon')^n < y$. Set $N = \max\{N_1, N_2\}$. Then for any $n \geq N$, we have

$$(1-\varepsilon')^n < y < (1+\varepsilon')^n.$$

Taking the nth root of both sides yields

$$1 - \varepsilon' < \sqrt[n]{y} < 1 + \varepsilon'$$

whence

$$|\sqrt[n]{y} - 1| < \varepsilon' \leq \varepsilon$$

as required. □

2.2.3 Divergent sequences

The sequence $\{a_n\}_{n\in\mathbb{N}}$ is convergent if

$$\exists L \in \mathbb{R} s.t. \forall \varepsilon > 0 \; \exists N \in \mathbb{N} s.t. \forall n \geq N, \; |a_n - L| < \varepsilon.$$

Hence the sequence is *divergent* if

$$\forall L \in \mathbb{R} \; \exists \varepsilon > 0 s.t. \forall N \in \mathbb{N} \; \exists n \geq N s.t. |a_n - L| \geq \varepsilon.$$

Example 2.9. Show that the sequence $a_n = (-1)^n$, $n \geq 0$, is divergent. (This is just the sequence $\{1, -1, 1, -1, \cdots\}$.)

We need to show that no real number a is the limit of this sequence.

First suppose $a \neq \pm 1$. Let $\varepsilon = \min\{|a-1|, |a+1|\}$, the minimum distance between a and any term of the sequence. Then for this particular value of ε, the inequality $|a - a_n| < \varepsilon$ fails for all $n \in \mathbb{N}$. Therefore it is impossible to satisfy the condition; therefore a is not the limit.

Next, suppose $a = 1$. (The case for $a = -1$ is similar and left as an exercise.) Choose $\varepsilon = 1$. Then the inequality $|a - a_n| < \varepsilon$ fails for every odd n, since when n is odd, $|a - a_n| = |1 - (-1)| = 2 \geq \varepsilon$. But no matter how we choose $N \in \mathbb{N}$, there will be an odd number $n \geq N$ (such as $2N + 1$) so it is impossible to satisfy the condition. Hence 1 is not the limit of the sequence.

(The case for $a = -1$ is similar and left as an exercise; it isn't a limit either.)

Since no number is a limit, this sequence diverges.

This example reveals the key property of convergent sequences, as stated in the next section.

There is one kind of divergent sequence that is fairly interesting.

Definition 2.10. Let $\{a_n\}$ be a sequence of real numbers. We say $\{a_n\}$ diverges to infinity, which we write as $a_n \to \infty$ or $\lim_{n\to\infty} a_n = \infty$, if

$$\forall M \in \mathbb{N} \; \exists N \in \mathbb{N} s.t. \forall n \geq N \; a_n > M.$$

Similarly, we say that $\{a_n\}$ diverges to negative infinity, written $a_n \to -\infty$ or $\lim_{n\to\infty} a_n = -\infty$, if

$$\forall M \in \mathbb{N} \; \exists N \in \mathbb{N} s.t. \forall n \geq N \; a_n < -M.$$

Remark 2.11. Such a sequence does not *converge* to $\pm\infty$ – notice that we can't even apply the definition of convergence with the limit value $L = \infty$ because ∞ is not a real number. So this notation, though kind of inconsistent, is just a convenient extension but should not cause confusion.

Example 2.12. Prove that $a_n = n$ diverges to ∞, while $b_n = -2^n$ diverges to $-\infty$, and $c_n = (-2)^n$ diverges, but neither to ∞, nor to $-\infty$.

Let $a_n = n$. Let $M \in \mathbb{N}$. Set $N = M$. Then for every $n \geq N$, we have $a_n = n \geq N = M$. $\square$

Let $b_n = -2^n$. Let $M \in \mathbb{N}$. Note that for every $n \geq 1$, $n < 2^n$ (proof by induction, exercise). Set $N = M$. Then for every $n \geq N$, we have $b_n = -2^n < -n \leq -N = -M$. $\square$

Let $c_n = (-2)^n$. Then c_n does not diverge to ∞ since with $M = 0$, for every $N \in \mathbb{N}$, there is an odd number $n \geq N$, for which we have $c_n = (-2)^n = -2^n < 0 = M$. (Exercise: prove the rest.)

So there are sequences, like $\{(-2)^n\}$ and $\{(-1)^n\}$, that diverge, but not in any direction. We will need other properties and words to describe interesting features of such sequences (like : are they bounded, do they have convergent subsequences?).

Example 2.13. Prove that $\lim_{n\to\infty} \frac{2n^3 - 2n^2 + 4n}{5n^2 - 4n + 10} = \infty$.

We begin by simplifying the expression (like in Example 2.6), which yields

$$\frac{2n^3 - 2n^2 + 4n}{5n^2 - 4n + 10} = \frac{2}{5}n - \frac{2}{25} + \frac{42n}{25(5n^2 - 4n + 10)}$$

and so we see that

$$\frac{2}{5}n - \frac{2}{25} + \frac{42n}{25(5n^2 - 4n + 10)} \geq \frac{2}{5}n - 1$$

(We could have chosen from zillions of other approximations! The point is just to choose one that makes it obvious that the sequence grows without bound, to make the rest of the proof easy.)

Proof. Let $M \in \mathbb{N}$. Choose $N = 5(M + 1)$. Then for all $n \geq N$, we have $n \geq N \geq 5(M + 1)/2$ so

$$\frac{2n^3 - 2n^2 + 4n}{5n^2 - 4n + 10} \geq \frac{2}{5}n - 1 \geq \frac{2}{5}(5(M + 1)/2) - 1 = M.$$

□

2.2.4 Uniqueness of the limit

Theorem 2.14. *Let* $\{a_n\}_{n\geq 1}$ *be a sequence. If* $\lim_{n\to\infty} a_n$ *exists, then it is unique.*

Proof. Suppose to the contrary that the sequence $\{a_n\}_{n\geq 1}$ had two limits, call them a and b. If $a \neq b$, then set $\varepsilon = \frac{1}{2}|b - a| > 0$.

Since the sequence converges to a by hypothesis, there is some integer N such that for every $n \geq N$, $|a - a_n| < \varepsilon$.

Since the sequence converges to b by hypothesis, there is some integer M such that for every $n \geq M$, $|b - a_n| < \varepsilon$.

Let $N' = \max\{N, M\}$. Then if $n \geq N'$, we have both $n \geq N$ and $n \geq M$, so we conclude that for any $n \geq N'$:

$$\begin{aligned} |b - a| &\leq |b - a_n| + |a_n - a| \quad \text{by the triangle inequality} \\ &= |b - a_n| + |a - a_n| \\ &< \varepsilon + \varepsilon \\ &= |b - a| \end{aligned}$$

This is a contradiction (we have just proven that $|b-a| < |b-a|$, which is false), so we conclude that $a = b$ and thus the limit, if it exists, is unique. □

As a consequence of the theorem, we refer to *the* limit of a convergent sequence, and we can see more intuitively why a sequence like $(-1)^n$ is divergent.

2.2.5 Limits and intervals

Proposition 2.15. *Let $c < d$ be real numbers and let $\{x_n\}_{n\in\mathbb{N}}$ be a convergent sequence with limit x such that for all $n \in \mathbb{N}$ we have*

$$x_n \in [c, d].$$

Then $x \in [c, d]$.

Proof. We are given that for all $n \in \mathbb{N}$, $c \leq x_n \leq d$. We proceed by contradiction: let us assume $x \notin [c, d]$. This means that either $x > d$ or $x < c$.

If $x > d$ then set $\varepsilon = x - d$. Then for all $n \in \mathbb{N}$, $x_n \leq d$ so $|x - x_n| \geq x - d = \varepsilon$. We conclude that $\{x_n\}_{n\in\mathbb{N}}$ cannot converge to x, a contradiction. The argument for $x < c$ is similar.

We conclude therefore that $x \in [c, d]$. □

Important! This theorem does not hold if we replace the closed interval $[c, d]$ with an open interval (c, d). For example, the sequence

$$\{\frac{1}{n}\}_{n\geq 1}$$

is entirely contained in the open interval $(0, 1)$ but its limit is 0, which is not contained in $(0, 1)$.

2.2.6 Exercises

1. For each of the following sequences, decide if they are convergent. If it is, find the limit, and then prove using the definition of the limit of the sequence that your limit is correct:

   (a) $a_n = (-1)^n \frac{1}{n}$, $n > 0$

   (b) $a_n = \frac{2n+3}{5n^2}$, $n > 0$

   (c) $a_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}$, $n \geq 0$

   (d) $a_n = n$

   (e)
   $$a_n = \begin{cases} 1 & \text{if n is even} \\ \frac{1}{n} & \text{if } n \text{ is odd} \end{cases}$$

   (f)
   $$a_n = \begin{cases} 0 & \text{if n is even} \\ \frac{1}{n} & \text{if } n \text{ is odd} \end{cases}$$

2. Prove that if a sequence $\{a_n\}$ satisfies, for some $a \in \mathbb{R}$, the following condition:

$$\exists N \in \mathbb{N}, \quad \text{such that } \forall \varepsilon > 0, \forall n \geq N, \quad \text{we have } |a_n - a| < \varepsilon$$

then the sequence is eventually constant, that is, there is some N such that for all $n \geq N$, $a_n = a$. *Moral: the order of the things you say in the definition is crucial for the meaning. You can swap the order of two "for all" statements but you can't usually move around a "there exists".*

3. Prove that if $a_n \geq c$ for all $n \in \mathbb{N}$ and if $\{a_n\}_{n\in\mathbb{N}}$ is convergent then $\lim_{n\to\infty} a_n \geq c$.

4. Given an example of a convergent sequence $\{a_n\}_{n\in\mathbb{N}}$ and a number c such that for all $n \in \mathbb{N}$ we have $a_n > c$ but for which it is NOT true that $\lim_{n\to\infty} a_n > c$.

5. Prove that Proposition 2.15 holds if you replace the closed and bounded interval $[c, d]$ with the closed and unbounded interval $[c, \infty)$ or $(-\infty, d]$.

6. Prove that $\lim_{n\to\infty} \frac{1}{n} = 0$.

7. Prove that $\lim_{n\to\infty} \frac{3n^2+2n+1}{n^2-7} = 3$.

8. Prove that $\lim_{n\to\infty} y^n = 0$ if $0 < y < 1$. (Hint: see a Corollary we proved earlier using induction)

9. Prove that $\lim_{n\to\infty} y^n = \infty$ if $y > 1$.

10. Prove that $\lim_{n\to\infty} y^n = 1$ if $y = 1$.

11. Prove that $\{y^n\}$ diverges if $y < -1$, but does not diverge to ∞ or $-\infty$.

12. Prove that $\lim_{n\to\infty} |-3 + \frac{1}{n}| = 3$.

13. Prove that $\lim_{n\to\infty} |1 + (-1)^n \frac{1}{n}| = 1$.

2.3 Proving convergence of sequences by other means

2.3.1 Algebra with limits of convergent sequences

Things would be really painful if we always had to go back to the definition to prove that a sequence was convergent. The following theorem, which hopefully feels intuitively true to you, allows us to conclude the convergence of a sequence, and even find its limit, by knowing the convergence of some more basic sequences.

This theorem also finally tells us how to add and multiply real numbers!

Theorem 2.16. *Suppose* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$. *Then the following are true:*

1. *for any* $c \in \mathbb{R}$, $\lim\limits_{n\to\infty} (ca_n) = ca$

2. $\lim\limits_{n\to\infty} (a_n + b_n) = a + b$

3. $\lim\limits_{n\to\infty} (a_nb_n) = ab$

4. if $b \neq 0$ then for some $N \in \mathbb{N}$, $\{\frac{a_n}{b_n}\}_{n \geq N}$ is a sequence and $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$

5. if $\exists N \in \mathbb{N}$ such that for all $n \geq N$ we have $a_n \geq b_n$ then $a \geq b$

Proof. 1. If $c = 0$ then the sequence is the constant sequence $\{0\}$ so converges to 0, and we're done. Otherwise, assume $c \neq 0$ and let $\varepsilon > 0$. *(We want to prove an inequality of the form $|ca_n - ca| < \varepsilon$; and we see immediately how to relate this to the inequality we may assume is true due to the convergence of $\{a_n\}$, namely:)* Since $\{a_n\}$ converges to a, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a - a_n| < \varepsilon|c|^{-1}$, whence

$$|ca_n - ca| = |c||a - a_n| < |c|(\varepsilon|c|^{-1}) = \varepsilon$$

as required.

2. (exercise)

3. We need to arrive at the inequality $|a_nb_n - ab| < \varepsilon$. We use the method from the homework:

$$\begin{aligned} |a_nb_n - ab| &= |a_nb_n - ab_n + ab_n - ab| \\ &\leq |a_nb_n - ab_n| + |ab_n - ab| \\ &= |b_n||a_n - a| + |a||b_n - b| \end{aligned}$$

We want this to come out to less than ε; a standard tactic is therefore to figure out why we can make each summand (that is, $|b_n||a_n - a|$ and $|a||b_n - b|$) less than $\varepsilon/2$, respectively. So let's do that:

- Since $\{b_n\}$ is convergent, it is bounded, so there is some $B > 0$ such that $|b_n| \leq B$ for all n. Since $\{a_n\}$ converges to a, there is some N_1 such that for all $n \geq N_1$, $|a - a_n| < (\varepsilon/2)B^{-1}$. So for $n \geq N_1$, we have $|b_n||a_n - a| < B((\varepsilon/2)B^{-1}) = \varepsilon/2$. Good.
- If $a = 0$ then $|a||b_n - b| = 0 < \varepsilon/2$, for any n. Otherwise, $|a| > 0$ and so $|a|^{-1} > 0$. Since $\{b_n\}$ converges to b, there is some $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$ we have $|b - b_n| < (\varepsilon/2)|a|^{-1}$. So it follows that for any $n \geq N_2$ we have $|a||b_n - b| < |a|(\varepsilon/2)|a|^{-1} = \varepsilon/2$. Good.

So our proof goes as follows. Let $\varepsilon > 0$. Choose N_1 and N_2 as per above. Then for any $n \geq \max\{N_1, N_2\}$, we have, as per the derivation above, that

$$|a_nb_n - ab| \leq |b_n||a_n - a| + |a||b_n - b| < \varepsilon/2 + \varepsilon/2 < \varepsilon$$

whence the sequence $\{a_nb_n\}$ converges to ab, as required.

4. (exercise)

5. (exercise)

□

Remark 2.17. So if we have defined real numbers as limits of sequences of rational numbers (such as sequences of decimal expansions), then the sum of two real numbers is given as the limit of the sequence obtained by adding the two sequences, term by term. Same for multiplication.

A way of thinking of this: so we realize we can't add (or subtract) real numbers properly, because wherever we start, our first digit could be wrong. But the theorem promises that the error you get is never very large, and diminishes as you repeat the process with more and more decimal places.

Example 2.18. Find $\lim_{n\to\infty} \frac{3n^2+4n+2}{2n^2+n+1}$.

We know that $\lim_{n\to\infty} \frac{1}{n} = 0$. We have

$$\lim_{n\to\infty} \frac{3n^2+4n+2}{2n^2+n+1} = \lim_{n\to\infty} \frac{3+4/n+2/n^2}{2+1/n+1/n^2} = \frac{3}{2}$$

by applying Theorem 2.16 repeatedly.

Example 2.19. Let $-1 < r < 1$. Then

$$\lim_{n\to\infty}(1+r+\cdots+r^n) = \lim_{n\to\infty} \frac{1-r^{n+1}}{1-r} = \frac{1}{1-r}$$

since $\lim_{n\to\infty} r^{n+1} = 0$ for $r \in (-1,1)$.

2.3.2 Convergent sequences are bounded

Theorem 2.20. *If $\{a_n \mid n \geq 1\}$ is a convergent sequence, then it is bounded.*

The idea: Intuitively, the definition implies that, excepting perhaps finitely many terms at the start of the sequence, the rest of the terms a_n are quite close to a; and so there's no way that you can have arbitrarily large elements in the sequence.

Proof. Let $a = \lim_{n\to\infty} a_n$. Let $\varepsilon = 1$; then there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, $|a - a_n| \leq 1$. This means that for all $n \geq N$ we have

$$a-1 < a_n < a+1.$$

But what about the first N terms of the sequence, namely $a_0, a_1, \cdots, a_{N-1}$? The above bounds may or may not hold for these elements. But that's OK: since there are only finitely many of them, we can set

$$M = \max\{a_0, a_1, \cdots, a_{N-1}\}$$

and

$$m = \min\{a_0, a_1, \cdots, a_{N-1}\}$$

so that we have, for all n such that $0 \leq n < N$,

$$m \leq a_n \leq M.$$

Finally, we put them together: set

$$L = \min\{m, a-1\}, \quad U = \max\{M, a+1\}$$

so that for all $n \in \mathbb{N}$ we have

$$L \leq a_n \leq U.$$

We conclude that the (entire) sequence is bounded. □

Alternate proof; optional. A different way to show that a set is bounded is to prove that there is a number $B > 0$ such that for all $n \in \mathbb{N}$, we have $|a_n| \le B$. This is equivalent to saying it is bounded above and below, since $|a_n| \le B$ iff $-B \le |a_n| \le B$.

Let $a = \lim_{n\to\infty} a_n$. Let $\varepsilon = 1$; then there exists some $N \in \mathbb{N}$ such that for all $n \ge N$, $|a - a_n| \le 1$. Then by the triangle inequality, we have that for all $n \ge N$,

$$|a_n| \le |a_n - a| + |a| = 1 + |a|$$

Set $M = \max\{|a_0|, |a_1|, \ldots, |a_{N-1}|\}$, so that for all n with $0 \le n < N$ we have

$$|a_n| \le M.$$

Therefore, setting $B = \max\{M, 1 + |a|\}$ we have that for all $n \in \mathbb{N}$,

$$|a_n| \le B$$

as required. □

We most commonly use this property to show that something is divergent, as follows.

Example 2.21. Show that the sequence given by $a_n = 1 + (-1)^n n$, for $n \ge 0$, is divergent.

Solution (left as an exercise in class): We claim that the set $S = \{1 + (-1)^n n \mid n \ge 0\}$ is not bounded. Namely, let $B \in \mathbb{R}$. We have that

$$|a_n| = |1 + (-1)^n n| \ge |(-1)^n n| - |1| = |n| - 1 = n - 1$$

so when we choose (by the Archimedean principle) a natural number $n > B + 1$, we conclude that $|a_n| > B$. Hence B is not an upper bound on $\{a_n\}$ whence the sequence is unbounded, hence divergent by the Theorem.

Example 2.22. Show that $\{a_n\}_{n\in\mathbb{N}} = \{(-1)^n n - \sin(n)\}_{n\in\mathbb{N}}$ is divergent.

We note that

$$|(-1)^n n - \sin(n)| \ge |(-1)^n n| - |\sin(n)| \ge n - 1$$

since $|(-1)^n n| = n$ and $|\sin(n)| \le 1$ for all n. Thus $\forall B \in \mathbb{N}$, set $N = B + 1$; then $|a_N| > B$ so the sequence is not bounded. Hence, by the theorem, it is divergent.

2.3.3 Bounded monotone sequences are convergent

Theorem 2.20 tells us that *if* a sequence is convergent *then* it is bounded. The converse is clearly false in general, since for example the sequence $\{(-1)^n\}_{n\ge 0}$ is bounded but divergent. However, there is a circumstance in which boundedness *does* imply convergent: when your sequence is always increasing (or always decreasing). We first define this concept, and then state the theorem.

Definition 2.23. A sequence $\{a_n\}_{n\in\mathbb{N}}$ is called *increasing* if for every $n \in \mathbb{N}$, we have $a_n \le a_{n+1}$. It is called *decreasing* if $\forall n \in \mathbb{N}$, $a_n \ge a_{n+1}$.[1] A sequence satisfying either of these conditions is called *monotone*.

[1] Some authors, like Trench [T], use "nondecreasing" and "nonincreasing" for these two concepts and reserve "increasing" and "decreasing" for sequences where the inequalities are strict. We will stick to this simpler definition and use "strictly increasing" or "strictly decreasing" if we need to discuss cases with strict inequalities.

Example 2.24. The sequence $\{1/n\}_{n\geq 1}$ is decreasing. The constant sequence $\{1\}_{n\geq 1}$ is monotone (in fact both increasing and decreasing!). The sequence $\{n\}_{n\geq 1}$ is increasing.

Theorem 2.25. *If a sequence $\{a_n\}_{n\in\mathbb{N}}$ is monotone and bounded then it is convergent.*

Proof. There are two cases to consider; let's prove it for the case that $\{a_n\}_{n\in\mathbb{N}}$ is decreasing.

Since the set $S = \{a_n \mid n \in \mathbb{N}\}$ is bounded, it is bounded below and so has an infimum; set $a = \inf(S)$. We claim that $a = \lim_{n\to\infty} a_n$.

Let $\varepsilon > 0$. Since $a = \inf(S)$, there exists an element $x \in S$ such that $a \leq x < a+\varepsilon$. Now $x \in S$ so in fact $x = a_N$ for some $N \in \mathbb{N}$. Let $n \geq N$; then since $\{a_n\}_{n\in\mathbb{N}}$ is decreasing, we have

$$a_N \geq a_{N+1} \geq \cdots \geq a_{n-1} \geq a_n.$$

Since for any $n \geq N$ we have $a_n \in S$, we know $a \leq a_n$. Therefore we have that for all $n \geq N$,

$$a - \varepsilon < a \leq a_n \leq a_N < a + \varepsilon$$

whence we conclude that

$$|a - a_n| < \varepsilon$$

(by exercises in the DGD and work done in class). Hence $a = \lim_{n\to\infty} a_n$, as required. □

2.3.4 Exercises

1. Prove part (b) of Theorem 2.16.
2. Prove part (d) of Theorem 2.16. This one is tougher. Remove some distractors by starting with the case that $b_n \neq 0$ for all n; then prove that if the limit isn't 0, there is some N after which $b_n \neq 0$, so you can do the general case.
3. Prove part (e) of Theorem 2.16. For the last part: you may proceed by contradiction. If $a < b$ then prove that necessarily there is an N such that for $n \geq N$, $a_n < \frac{a+b}{2} < b_n$.
4. Prove the Squeeze Theorem: Suppose $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences such that:
   - $\{a_n\}$ and $\{c_n\}$ converge to the same limit L, and
   - for some $N \in \mathbb{N}$, we have that for all $n \geq N$, $a_n \leq b_n \leq c_n$.

   Prove that $\{b_n\}$ is convergent and has the same limit L.
5. Suppose $\{a_n\}$ is a sequence of real numbers and $\lim_{n\to\infty} |a_n| = 0$. Prove that $\lim_{n\to\infty} a_n = 0$.
6. Given an example of a divergent sequence $\{a_n\}$ such that the sequence $\{|a_n|\}$ is convergent.
7. Prove that if $\{a_n\}_{n\geq 0}$ is a convergent sequence, then so is the sequence labeled $\{a_{n+1}\}_{n\geq 0}$, and they have the same limit.
8. Let $a_1 = \frac{3}{2}$ and $a_{n+1} = 2 - \frac{1}{a_n}$ for $n \geq 1$.
   - Show that $0 \leq a_n \leq 2$ for all $n \geq 1$ (hint: use induction).

- Show that a_n is a monotone sequence. (Hint: calculate the difference $a_{n+1} - a_n$ and think of the graph of the function $y = x^2 - 2x + 1$.)
- Conclude that $\{a_n\}$ is convergent.
- Call the limit a. Using the previous exercise, and the recursive formula for a_n, find a.

9. Find the limits of the following sequences and use the theorem of the algebra of convergent sequences to prove your answer:

   (a) $a_n = 3(\frac{1}{2^n}) - (0.2)^n$

   (b) $a_n = \frac{12}{5n}$

   (c) $a_n = \frac{1}{n2^n}$

10. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence.

   (a) Prove that if $\{a_n\}_{n\in\mathbb{N}}$ converges then $\{|a_n|\}_{n\in\mathbb{N}}$ converges.

   (b) Give an example of a divergent sequence $\{a_n\}_{n\in\mathbb{N}}$ such that $\{|a_n|\}_{n\in\mathbb{N}}$ converges.

   (c) Prove that if $\{|a_n|\}_{n\in\mathbb{N}}$ converges to 0, then so does $\{a_n\}_{n\in\mathbb{N}}$.

2.4 Subsequences and the Bolzano-Weierstrass Theorem

Definition 2.26. Let $\{a_n\}_{n\geq 0}$ be a sequence. A *subsequence* of $\{a_n\}$ is a sequence

$$a_{n_1}, a_{n_2}, a_{n_3}, \cdots$$

where $n_k \in N$ and these indices satisfy

$$n_1 < n_2 < n_3 < \cdots$$

(that is, the indices are themselves a strictly increasing sequence of natural numbers). We denote such a subsequence by $\{a_{n_k}\}$.

Example 2.27. The sequence $\{(-1)^n/n\}$ contains as a subsequence $\{1/(2n)\}$.

Example 2.28. The sequence $\{n^2\}_{n\geq 0}$ contains the subsequence $\{n^2\}_{n\geq 45}$.

A key theorem about sequences is the following.

Theorem 2.29 (Bolzano-Weierstrass Theorem)**.** *Every bounded sequence contains a convergent subsequence.*

Remark 2.30.

- The theorem doesn't say that every bounded sequence is convergent — just that you could (somehow) pick out a subsequence that does converge.
- The conclusion of the theorem is false for an unbounded sequence (think about $\{n\}_{n\in\mathbb{N}}$ for example).
- The theorem is obvious for a divergent bounded sequence like $\{(-1)^n\}_{n\in\mathbb{N}}$ (look at the subsequence of even terms, or of odd terms), and it's obvious for convergent sequences (since a sequence is a subsequence of itself).

- The theorem is NOT OBVIOUS, and absolutely fascinating, for divergent bounded sequences like $\{\sin(n)\}_{n\in\mathbb{N}}$.

To prove it, let's first prove a lemma that's more specific, and in fact much stronger, than the theorem.

Lemma 2.31. *Every sequence of real numbers contains a monotone subsequence.*

Proof. We build the subsequence carefully. This argument is particularly elegant.

Let $S = \{a_n \mid \forall m > n, \text{ we have } a_m \geq a_n\}$ be the set of all elements of the sequence which have the property that they are less than or equal to all remaining elements. (This set might typically not have any elements in it, of course.)

If S is infinite, then we're done, since the sequence consisting of all the elements of S, in order, is a subsequence of $\{a_n\}$.

Otherwise, S is finite (or empty, which is also finite). Since S contains only finitely many elements of the sequence, there is some n_1 such that for all $n \geq n_1$, $a_n \notin S$. Since $a_{n_1} \notin S$, there must exist some $m = n_2 > n_1$ such that $a_{n_2} < a_{n_1}$. Since $n_2 > n_1$, we know $a_{n_2} \notin S$; hence by definition there is some $m = n_3 > n_2$ such that $a_{n_3} < a_{n_2}$. Continuing in this way, we construct a (strictly) decreasing subsequence of $\{a_n\}$, as required. □

With this lemma in hand, we can prove the Bolzano-Weierstrass theorem easily.

Proof of Bolzano-Weierstrass. Let $\{a_n\}$ be a bounded sequence. By the Lemma, it has a subsequence $\{a_{n_k}\}$ which is monotone. The sequence $\{a_{n_k}\}$, being both monotone and bounded, is convergent by Theorem 2.25. □

Remark 2.32. That's a pretty short proof, but only because we did all the work previously.

Another way of thinking about this theorem: that every bounded infinite set has an *acculumlation point*, that is, a point such that in every interval around that point, there are infinitely many elements of the set.

2.4.1 An alternate proof of Bolzano-Weierstrass (optional)

Theorem 2.33 (Bolzano-Weierstrass Theorem : Alternate Direct Proof)**.** *Every bounded sequence contains a convergent subsequence.*

Proof. Let $\{a_n\}_{n\in\mathbb{N}}$ be a bounded sequence. We create a subsequence $\{a_{nk}\}_{k\in\mathbb{N}}$ inductively.

Base case: Since $\{a_n\}_{n\in\mathbb{N}}$ is bounded, there exist $x_0 < y_0$ such that for all $n \in \mathbb{N}$, $x_0 \leq a_n \leq y_0$. Let $a_{n_0} = a_0$, which lies in the interval $[x_0, y_0]$.

Inductive hypothesis: Suppose for some $k \geq 0$ we have constructed a subinterval $[x_k, y_k]$ such for for infinitely many values of n, $a_n \in [x_k, y_k]$, and suppose also that we have chosen $n_k \in \mathbb{N}$ so that $n_k > n_{k-1}$ and so that $a_{n_k} \in [x_k, y_k]$.

Induction step: Divide the interval $[x_k, y_k]$ into two halves: $[x_k, \frac{1}{2}(x_k + y_k)]$ and $[\frac{1}{2}(x_k + y_k), y_k]$. Since there are (by the inductive hypothesis) infinitely many values of n for which $a_n \in [x_k, y_k]$, at least one of these two intervals must have the same property: that there are infinitely many values of n for which a_n is in that interval[2]. Denote its endpoints by $x_{k+1} < y_{k+1}$. Since there are infinitely many values of n for which $a_n \in [x_{k+1}, y_{k+1}]$, we can choose an integer $n_{k+1} > n_k$ for which $a_{n_{k+1}} \in [x_{k+1}, y_{k+1}]$.

Since in the induction step we have constructed everything required for the next case $(k+1)$ of the inductive hypothesis, we deduce that by the principle of mathematical induction, this process will continue (forever).

We have thus constructed an increasing sequence x_k, which is bounded above by y_0 and hence is convergent, say to limit x. We also constructed a decreasing sequence y_k which is bounded below by x_0, and hence is convergent, say to limit y. Since $x_k < y_k$ for all $k \in \mathbb{N}$, we have (exercise) that $x \leq y$.

We claim they are equal. For suppose to the contrary that $x < y$. Let $\varepsilon = y - x > 0$. Since $\{x_k\} \to x$, $\{y_k\} \to y$ and $\{y_k - x_k\} \to 0$, we can find K such that for all $k \geq K$ we have

$$|x - x_k| < \varepsilon/3, |y - y_k| < \varepsilon/3, \quad |y_k - x_k| < \varepsilon/3$$

whence by the triangle inequality, $|x - y| < \varepsilon$, a contradiction. So $x = y$.

We claim that our subsequence is convergent and that x is its limit.

Let $\varepsilon > 0$. Choose $K \in \mathbb{N}$ so that for all $k \geq K$, we have $|x_k - y_k| < \varepsilon/2$ and $|x - x_k| < \varepsilon/2$. Then

$$|a_{n_k} - x| \leq |a_{n_k} - x_k| + |x_k - x| \leq |y_k - x_k| + |x_k - x| < \varepsilon$$

(where we have used that since $a_{n_k} \in [x_k, y_k]$, $|a_{n_k} - x_k| \leq |y_k - x_k|$), as required. □

2.4.2 Cauchy sequences (optional)

Definition 2.34. A sequence $\{a_n\}$ is called *Cauchy* if it has the following property:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall m, n \geq N, \text{ we have } |a_n - a_m| < \varepsilon.$$

Lemma 2.35. *If $\{a_n\}$ is convergent, it is Cauchy.*

Proof. Let $\varepsilon > 0$. Since $\{a_n\}$ is convergent, say with limit a, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a - a_n| < \varepsilon/2$. Hence for any $n \geq N$ and $m \geq N$ we have

$$|a_m - a_n| \leq |a_n - a| + |a - a_m| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

as required. □

Example 2.36. Insert nice example.

[2] If that feels difficult to trust: explain what it would mean to say that this is false.

One might ask: are there any Cauchy sequences that are *not* convergent? Answer: no, as per the following theorem.

Theorem 2.37. *If a sequence $\{a_n\}$ is Cauchy, then it is convergent.*

Proof. Suppose $\{a_n\}$ is Cauchy. Then (by making particular choices) there exists some $N \in \mathbb{N}$ such that for all $m \geq N$, $|a_m - a_N| < 1$, whence we conclude that $\forall m \geq N$, $|a_m| \leq |a_N| + 1$. Arguing as in the proof of Theorem 2.20, we see that $\{a_n\}$ is bounded.

Hence by the Bolzano-Weierstrass Theorem, $\{a_n\}$ has a convergent subsequence, say $\{a_{n_k}\}$. Denote the limit of this subsequence by a; we claim that $a = \lim_{n\to\infty} a_n$.

Namely, let $\varepsilon > 0$. Since $\{a_n\}$ is Cauchy, there is some $N' \in \mathbb{N}$ such that for all $m, n \geq N'$, we have $|a_m - a_n| < \varepsilon/2$.

Since $\{a_{n_k}\}$ converges to a, there exists some $M \in \mathbb{N}$ such that for all $n_k \geq M$, we have $|a_{n_k} - a| < \varepsilon/2$.

Choose an index n_k of the subsequence which satisfies both $n_k \geq M$ and $n_k \geq N'$; this is possible because indices of a subsequence are unbounded. Then we have, for all $m \geq N'$, that

$$|a_m - a| \leq |a_m - a_{n_k}| + |a_{n_k} - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

as required. □

So an alternative way to prove that a sequence is convergent is to prove that it is Cauchy — and you can prove a sequence is Cauchy without knowing what the limit is! Hence it's a really useful criterion. Furthermore, it allows us to talk about "Cauchy sequences of rational numbers" which is the necessary ingredient in filling in the apparent circularity of the proof that all real numbers are the limits of all convergent sequences of rational numbers (instead, it's the set of all limits of all Cauchy sequences of rational numbers; together with a relation identifying when two numbers are equal even though their sequences look different).

2.4.3 Exercises

1. Prove that a decreasing sequence which is bounded below converges to its infimum.
2. Prove that any subsequence of a convergent sequence is convergent (to the same limit).
3. Prove that if a sequence contains two convergent subsequences such that the limits of these subsequences are not equal, then the original sequence is divergent.
4. Give an example of a sequence which is divergent yet has two subsequences which converge to the same limit.
5. Prove that if $\lim_{n\to\infty} a_n = a$ then $\lim_{n\to infty} a_{n+1} = a$ and $\lim_{n\to\infty} a_{2n+1} = a$.
6. Let $a_1 = 3/2$ and $a_{n+1} = 2 - a_n^{-1}$ for each $n \geq 1$.

   (a) Prove that $1 \leq a_n \leq 2$ for all $n \geq 1$.

(b) Prove that the sequence a_n is monotone. (Hint: calculate the difference $a_{n+1} - a_n$ and think of the graph of the function $y = x^2 - 2x + 1$.)

(c) Why is the sequence a_n convergent?

(d) Let a be its limit. Given that $\lim_{n\to\infty} a_{n+1} = a$ also, use the recursive formula for a_{n+1} to determine the value of a.

7. Prove that $\lim_{n\to\infty} n^{1/n} = 1$. Hint: write $n^{1/n} = 1 + a_n$ for some unknown sequence a_n. Then show that $(1 + a_n)^n \geq \frac{n(n-1)}{2} a_n^2$ to see that $a_n \to 0$.

8. Suppose $\{a_n\}_{n\in\mathbb{N}}$ is a sequence such that the subsequences $\{a_{2n}\}_{n\in\mathbb{N}}$ and $\{a_{2n+1}\}_{n\in\mathbb{N}}$ both converge to the same limit. Prove that $\{a_n\}_{n\in\mathbb{N}}$ converges.

Chapter 3

Functions

In Calculus, we are interested in functions whose domain and range are subsets of $\mathbb{R}$. Throughout this chapter let U be an interval: open, or closed, or sem-open; then we consider functions $f\colon U \to \mathbb{R}$ with domain U and image in $\mathbb{R}$. In Calculus, the functions we are most interested in have domains which are unions of a finite number of such intervals.

3.1 Functions, limits and continuity

3.1.1 Definition of a function, and limit of a function

If $\{x_n\}_{n\in\mathbb{N}}$ is a sequence contained in the domain of a function f, then we can create a new sequence by applying f:

$$\{f(x_n)\}_{n\in\mathbb{N}}.$$

For example, if we start with the sequence $\{1/n\}_{n\in\mathbb{N}}$ and our function is $f(x) = 2 + x + x^2$ then we obtain the sequence $\{2 + \frac{1}{n} + \frac{1}{n^2}\}_{n\in\mathbb{N}}$.

This observation allows us to define precisely what we mean by

$$\lim_{x\to c} f(x) = L,$$

which is the essential ingredient for Calculus, starting with the definition of the continuity of a function.

Definition 3.1. Let $f\colon U \to \mathbb{R}$ be a function and suppose $c \in U$ or else c is the limit of a sequence in U. We say that the limit of f as x approaches c is L,[1] and write

$$\lim_{x\to c} f(x) = L$$

if for all sequences $\{x_n\}_{n\in\mathbb{N}}$ with the following properties:

[1] If $c = \sup(U)$ or $c = \inf(U)$ then properly speaking we are only evaluating the left-hand or right-hand limit. If there is possibility of confusion (that is, if the domain of the function could be larger) it is better to write $\lim_{x\to c-} f(x)$ or $\lim_{x\to c+} f(x)$, respectively.

1. $\lim_{n\to\infty} x_n = c$
2. for all $n \geq 0$, $x_n \in U$
3. for all $n \geq 0$, $x_n \neq c$

we have

$$\lim_{n\to\infty} f(x_n) = L.$$

Example 3.2. Let

$$f(x) = \begin{cases} x-3 & \text{if } x \neq 7 \\ 2 & \text{if } x = 7. \end{cases}$$

What is $\lim_{x\to 7} f(x)$?

Solution: Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ which converges to 7, such that for all $n \in \mathbb{N}$, $x_n \neq 7$. Thus $f(x_n) = x_n - 3$. Then

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} (x_n - 3) = \lim_{n\to\infty} x_n - 3 = 7 - 3 = 4.$$

This holds for any such sequences, so we conclude

$$\lim_{x\to 7} f(x) = 4.$$

which has nothing to do with $f(7)$.

3.1.2 Left and right limits (optional)

Our definition of the limit requires us to consider *all* sequences converging to c, not just those coming from the right or the left. But we know from Calculus I that it suffices to look at one-sided limits. Let us resolve this apparent contradiction.

Definition 3.3. Let $f\colon U \to \mathbb{R}$ and let $c \in \mathbb{R}$.

First suppose there exists at least one sequence of elements of U which are less than c which converge to c. Then we say

$$\lim_{x\to c^-} f(x) = L$$

if for *every* sequence $\{x_n\}_{n\in\mathbb{N}}$ in $U \cap (-\infty, c)$ such that $\lim_{n\to\infty} x_n = c$, we have

$$\lim_{n\to\infty} f(x_n) = L.$$

This is called the left-hand limit of f as x approaches c. Similarly, if there is at least one sequence of elements of U greater than c which converges to c, then we say

$$\lim_{x\to c^+} f(x) = L$$

if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in $U \cap (c, \infty)$ such that $\lim_{n\to\infty} x_n = c$, we have

$$\lim_{n\to\infty} f(x_n) = L.$$

This is called the right-hand limit of f as x approaches c.

Lemma 3.4. *Let $f\colon U \to R$ and $c \in R$ be such that there exists at least one sequences in $U \cap (c, \infty)$ converging to c and at least one sequence in $U \cap (-\infty, c)$ converging to c. Then*

$$\lim_{x \to c} f(x) = L$$

if and only if

$$\lim_{x \to c^-} f(x) = L \quad \textit{and} \quad \lim_{x \to c^+} f(x) = L.$$

$\Rightarrow$. : there is nothing to prove. Under the hypothesis, all sequences converging to c give the same limit when you apply f; so in particular this is true of all sequences approaching from the left, or from the right.

$[\Leftarrow]$: Suppose $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in U converging to c, such that $x_n \neq c$ for all n. Let $G = \{n \in \mathbb{N} \mid x_n < c\}$. If G is finite, then there is some $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n > c$. Therefore by hypothesis, the sequence $\{f(x_n)\}_{n \geq N}$ converges to L, whence $\{f(x_n)\}_{n \in \mathbb{N}}$ converges to L.

Similarly, let $R = \{n \in \mathbb{N} \mid x_n > c\}$. If R is finite, then by the same argument we deduce that $\{f(x_n)\}_{n \in \mathbb{N}}$ converges to L.

If neither R nor G is finite, then each is infinite and defines a subsequence of $\{x_n\}_{n \in \mathbb{N}}$. These subsequences converge to c since the original sequence does; they consist of elements of U greater than (respectively, less than) c, so by hypothesis

$$\{f(x_n)\}_{n \in G} \to L \quad \text{et} \quad \{f(x_n)\}_{n \in R} \to L.$$

We claim that since $G \cup R = \mathbb{N}$, this implies that $\{f(x_n)\}_{n \in \mathbb{N}}$ converges to L (exercise). □

3.1.3 Dependence on the domain (optional)

Our definition of the limit of a sequence says that we must consider sequences in the entire domain of f. But shouldn't the limit really only depend on the sequences that are close to c to begin with?

Lemma 3.5. *Let $f\colon U \to R$ and $c \in U$. Suppose there is some $r > 0$ such that $(c - r, c + r) \subseteq U$. Let $V = (c - r, c + r)$; we may also consider $f\colon V \to \mathbb{R}$. Then the existence and value of $\lim_{x \to c} f(x)$ is independent of the choice of domain U or V.*

Proof. If $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in U converging to c (without taking value c), then there is some $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in (c - r, c + r)$. Thus since the limit of the sequence $\{f(x_n)\}_{n \in \mathbb{N}}$ coincides with the limit of the sequence $\{f(x_n)\}_{n \geq N}$, the result follows. □

On the other hand, this was not entirely silly. The function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

doesn't have a limit as x approaches $c = 0$, but if we restrict the function to the smaller domain $V = [0, \infty)$, we would say without hesitation that the limit exists!

Conclusion: there is a big difference between evaluating the limit on the interior of an interval versus on the boundary thereof.

3.1.4 Continuous functions

The most interesting case is often when $\lim_{x\to c} f(x) = f(c)$.

Definition 3.6. Let $f\colon U \to \mathbb{R}$ be a function and suppose $c \in U$. We say that the function f is continuous at c if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in U such that

$$\lim_{n\to\infty} x_n = c$$

the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ converges and we have

$$\lim_{n\to\infty} f(x_n) = f(c).$$

In this case, we write

$$\lim_{x\to c} f(x) = f(c).$$

Example 3.7. Consider $f(x) = x^2$ and let $c \in \mathbb{R}$. Let $\{x_n\}_{n\in\mathbb{N}}$ be any sequence converging to c. Then

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n^2 = c^2$$

by the algebra of convergent sequences, and since $f(c) = c^2$ we conclude that f is continuous at c. This works for all $c \in \mathbb{R}$ so f is continuous on $\mathbb{R}$.

Example 3.8. Consider the function $f\colon (0,1) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational;} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

We want to show it is discontinuous at every $c \in (0,1)$. To do that, it suffices to construct a sequence $\{x_n\}_{n\geq 1}$ converging to c such that $\{f(x_n)\}_{n\geq 1}$ does not converge to $f(c)$.

Say c is rational, so $f(c) = 0$. Let's construct a sequence of irrational numbers converging to c; we use the following STANDARD METHOD.

For each $n \geq 1$, choose x_n so that $c - \frac{1}{n} < x_n < c$ with the property that x_n is irrational; this is possible by the density theorem.

Claim: $\{x_n\}_{n\geq 1}$ converges to c.

Proof: Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ so that $N \geq 1/\varepsilon$. Then for all $n \geq N$ we have

$$|x_n - c| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

as required. $\square$

Good. Now since $f(x_n) = 1$ for every $n \geq 1$, the sequence $\{f(x_n)\}_{n\geq 1}$ is the constant sequence 1, which converges to $1 \neq f(c)$. Therefore f is not continuous at c.

Exercise: prove f is not continuous at any irrational c either.

3.1.5 Algebra of continuous functions

A continuous function has the property that for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in its domain, which converges to an element of its domain, we have

$$f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n).$$

The following theorem therefore is an immediate consequence of the Theorem of the Algebra of Convergent Sequences.

Theorem 3.9. *Suppose $f, g\colon U \to \mathbb{R}$ are two functions which are continuous at $c \in U$. Then*

1. *$f + g$ is continuous at c;*
2. *fg is continuous at c;*
3. *for any constant $r \in \mathbb{R}$, rf is continuous at c;*
4. *if $g(c) \neq 0$ then $\frac{f}{g}$ is defined on an open interval around c and is continuous at c;*
5. *if the image of f is contained in the domain of a function $h\colon V \to \mathbb{R}$ which is continuous at $f(c)$ then $h \circ f\colon U \to \mathbb{R}$ is continuous at c.*

Proof. The first three follow directly for the algebra of convergent sequences and the definition of continuity.

For the third: Suppose $g(c) \neq 0$. Then we claim there must be an open interval (a', b') containing c such that g is nonzero on that interval. For if not, then for every $n \geq 1$, we could find $x_n \in (c - 1/n, c + 1/n) \cap (a, b)$ such that $g(x_n) = 0$, whence $\{x_n\}_{n\geq 1} \to c$ but $\{g(x_N\}_{n\geq 1} \to 0 \neq g(c)$, a contradiction. So $\frac{f}{g}$ is defined on some open interval (a', b') containing c.

Now let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in (a', b') which converges to c. Then since

$$\lim_{n\to\infty} f(x_n) = f(c) \quad \text{and} \quad \lim_{n\to\infty} g(x_n) = g(c)$$

we deduce by the algebra of convergent sequences that

$$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \frac{f(c)}{g(c)},$$

as required.

For the last: Let $\{x_n\}_{n\in\mathbb{N}}$ be any sequence in U converging to c. Then since f is continuous, $\{f(x_n)\}_{n\in\mathbb{N}}$ is a sequence converging to $f(c)$, and it lies in V by hypothesis. Thus, since h is continuous at $f(c)$, we know that $\{h(f(x_n))\}_{n\in\mathbb{N}}$ converges to $h(f(c))$. Thus altogether we have show that the function $h \circ f$, which is $(h \circ f)(x) = h(f(x))$, is continuous at c. □

This theorem is quite useful. For example, since we know that $f(x) = x$ is a continuous function (exercise), it follows from the theorem that any polynomial function or rational function is also continuous (at every point in its domain).

We will not prove the following theorem in this course, but you may henceforth take it as known. Over the course of your mathematical career you will certainly prove it many times!

Theorem 3.10. *Let $r \in \mathbb{R}_{>0}$. The following functions on continuous on their domains:*

$$e^x, \ln(x), \sin(x), \cos(x), \arcsin(x), \arccos(x), \arctan(x), x^r.$$

Combining this theorem with the one about the algebra of continuous functions gives us the continuity of the vast majority of functions that we are familiar with.

3.1.6 Alternate criterion for continuity (optional)

There is another criterion for continuity, that does not make reference to sequences. We prove the following theorem in MAT2125.

Theorem 3.11. *A function $f\colon U \to \mathbb{R}$ is continuous at $c \in U$ if and only if*

$$\forall \varepsilon > 0\ \exists \delta > 0\ s.t.\ \forall x \in U\ |x - c| < \delta \ \Rightarrow\ |f(x) - f(c)| < \varepsilon.$$

3.1.7 Exercises

1. Let f be the function which is 1 at every integer and 0 otherwise. Find all c such that f is continuous, and all c at which f is not continuous. Prove your answer.

2. Show that the following functions are continuous at the given point, using the definition of continuity:

   (a) $f(x) = x$, $c \in \mathbb{R}$

   (b) $f(x) = 15x - 4$, $c = 5$

   (c) $f(x) = x^2$, $c = 5$

   (d) $f(x) = \frac{1}{x}$, $c = 5$

3. Use the theorem about the algebra of continuous functions to prove that the following functions are continuous at every point in their domain:

   (a) $f(x) = x^4 + x^3 + 1$

   (b) $f(x) = \frac{5x^4+x^2+1}{2x^3-x-1}$

4. Provide definitions, in terms of sequences, of the following concepts familiar from Calculus:

   (a) $\lim_{x\to\infty} f(x) = \infty$

   (b) $\lim_{x\to a} f(x) = \infty$: you'll need to restrict your sequences $\{x_n\}$ to have the property that $x_n \neq a$.

   (c) $\lim_{x\to\infty} f(x) = L$

5. Explain how you could construct a sequence of rational numbers converging to a number c using the STANDARD METHOD.

6. (a) (If you have not already done so) Prove that if $\{a_n\}_{n\in\mathbb{N}}$ is a sequence which converges to 0 and $\{b_n\}_{n\in\mathbb{N}}$ is a bounded sequence, then the sequence $\{a_nb_n\}_{n\in\mathbb{N}}$ converges to 0.

(b) Prove that if f is a continuous function such that $f(0) = 0$ and g is a function which is bounded on a interval $(-r, r)$ but is perhaps not continuous then the product function $(fg)(x) = f(x)g(x)$ is continuous in 0.

(c) Apply this result to deduce the continuity of the following function at 0:

$$h(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(Identify the functions f and g and explicitly verify that they satisfy the hypotheses in (b).)

(d) Give an example of a continuous function f satisfying $f(0) = 0$ and a function g such that $fg(x) = f(x)g(x)$ is not continuous at 0.

7. Decide if the following functions are continuous at c. Prove your answer using the tools proven in class.

(a) $f(x) = \begin{cases} x + 2 & \text{if } x > 2 \\ x^2 & \text{if } x \leq 2 \end{cases}, \ c = 2$

(b) $f(x) = \begin{cases} x + 2 & \text{if } x > 2 \\ x^2 - 1 & \text{if } x \leq 2 \end{cases}, \ c = 2$

(c) $f(x) = \begin{cases} x^2 + 3 & \text{if } x > 0 \\ -3 & \text{if } x \leq 0 \end{cases}, \ c = 0$

(d) $f(x) = \begin{cases} x^2 + 3 & \text{if } x > 0 \\ 3 - x^2 & \text{if } x \leq 0 \end{cases}, \ c = 0$

8. Let E be a nonempty set of real numbers which is bounded above and leet $s = \sup(E)$. Let f be the function

$$f(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Prove that if $c > s$ then f is continuous at c.

9. Let $V \subseteq U$ be intervals and let $f \colon U \to \mathbb{R}$ and $g \colon V \to \mathbb{R}$ be two functions such that for all $x \in V$, $f(x) = g(x)$. (Example: $f(x) = |x|$ with $U = \mathbb{R}$ and $g(x) = x$ with $V = [0, \infty)$.)

(a) Suppose $c \in V$ and that there is some $r > 0$ such that $(c - r, c + r) \subseteq V$. Prove that f is continuous at c iff g is continuous at c.

(b) Give an example of f, g, $V \subseteq U$ and $c \in V$ such that g is continuous at c but f is not.

10. Give a definition, in terms of sequences, of each of the following concepts familiar from Calculus:

(a) $\lim_{x\to\infty} f(x) = \infty$

(b) $\lim_{x\to a} f(x) = \infty$

(c) $\lim_{x\to\infty} f(x) = L$

3.2 Two major theorems about continuous functions

3.2.1 Intermediate value theorem

This theorem states a property that's "obvious" if we think of continuous functions as being ones whose graph you can draw without lifting your pencil. (But note that this "definition" of continuous functions isn't very accurate: for example, $f(x) = 1/x$ is continuous everywhere on its domain, but you have to lift your pencil to cross the vertical asymptote at $x = 0$. Even worse, there exist continuous functions whose graphs we can't even draw!)

That the following theorem holds for all functions which satisfy the complicated definition of continuity is quite remarkable, and can be taken as evidence that the definition chosen is exactly the right one (effectively: the idea that you can trace the graph without lifting your pencil and skipping a y-value is the statement of the Intermediate Value Theorem; so we came up with a definition of continuous functions for which that theorem would hold.).

Theorem 3.12 (Intermediate Value Theorem). *Let $f : U \to \mathbb{R}$ be a continuous function. Then for any $a, b \in U$ with $a < b$, and any $z \in \mathbb{R}$ between $f(a)$ and $f(b)$, there is some $s \in [a, b]$ such that $f(s) = z$.*

Example 3.13. The function $f(x) = x^2$ is continuous. If $a = -1$ and $b = 2$ then $f(a) = 1$ and $f(b) = 4$ and for any $z \in [1, 4]$ we can find $s \in [-1, 2]$ (in fact, in $[1, 2]$) such that $f(s) = z$.

Example 3.14. The function

$$f(x) = \begin{cases} x & \text{if } x > 1 \\ x - 1 & \text{if } x \leq 1 \end{cases}$$

is not continuous (exercise). Consider $a = 0$ and $b = 2$. We have $f(a) = -1$ and $f(b) = 2$, but if we choose $z = 1$ then there is no $s \in [0, 2]$ for which $f(s) = z$.

Remark 3.15. We sketch more examples to get a feel for what this theorem is saying, why it might be true, and how we might be able to prove it. From our graphs we realize that in fact we should expect there are many values of s for which $f(s) = z$. This creates a dilemma: how do we find a way to uniquely describe one of these values, enough to prove it works? A good idea is to try for the FIRST or LAST.

Proof of Intermediate Value Theorem. Let's assume that $f(a) < f(b)$; the proof for the case that $f(b) \leq f(a)$ is an exercise.

If $z = f(a)$ or $z = f(b)$ then we are done, since $s = a$ (respectively, $s = b$) is an inverse image (and could very well be the only such value!). Otherwise, $f(a) < z < f(b)$ and we define

$$T = \{u \in [a, b] \mid f(u) < z\}.$$

This set is nonempty since $a \in T$ and it is bounded above since every element $u \in T$ satisfies $u \leq b$. Therefore it has a supremum, which lies in $[a, b]$; write

$$s = \sup(T).$$

We will prove that $f(s) = z$. Construct a sequence $\{x_n\}_{n\geq 1}$ of elements of T converging to s, by the STANDARD METHOD, as follows. Since $s = \sup(T)$, for each $n \geq 1$ there is some $x_n \in T$ satisfying $s - \frac{1}{n} < x_n \leq s$; then $\lim_{n\to\infty}\{x_n\}_{n\geq 1} = s$. (Proof: exercise.)

Now consider $\{f(x_n)\}_{n\geq 1}$. By continuity, it converges to $f(s)$. Since for every $n \geq 1$ we have $f(x_n) < z$ we know that $f(s) \leq z$ (from Theorem 2.16).

Thus $s \neq b$. Construct a sequence $\{x'_n\}_{n\geq 1}$ of elements satisfying $s < x'_n < b$ converging to s, by the STANDARD METHOD. Since each x'_n is strictly greater than $s = \sup(T)$, we have $x'_n \notin T$, so $f(x'_n) \geq z$ for all $n \geq 1$, whence the limit $\lim_{n\to\infty} f(x'_n)$, which equals $f(s)$ by continuity, satisfies $f(s) \geq z$.

Thus $f(s) = z$. □

The Intermediate Value Theorem has a number of useful applications, since it promises the existence of a s with $f(s) = z$ (under appropriate hypotheses), even though it might be extremely difficult to actually find such a s.

Example 3.16. Consider $f(x) = x^5 + x^4 + 2x^3 + 3x + 1$. It has no rational roots (since neither 1 nor -1 is a root); how do we know it has any roots at all? Well, since f is continuous, and $f(0) = 1$ but $f(-1) = -4$, we conclude that there is at least one $z \in (-1, 0)$ such that $f(z) = 0 \in (-4, 1)$.

Remark 3.17. In fact, we could construct a sequence converging to this root, by bisecting the interval and comparing the signs of the endpoints of each half; but we saw in MAT1320 that it's far more efficient to use Newton's method when it applies (that is, when the function is differentiable, as this one is).

Example 3.18. You can use the intermediate value theorem to prove the existence of $\beta = \sqrt[n]{\alpha}$ for any $\alpha > 0$. Namely, the function $f(x) = x^n - \alpha$ is continuous, satisfies $f(0) < 0$. If $\alpha > 1$ then $f(\alpha) > 0$ and if $0 < \alpha \leq 1$ then $f(1) = 1 - \alpha \geq 0$. Therefore f must attain the value 0 at some point $\beta \in (0, \min\{1, \alpha\})$; and this β satisfies $\beta^n = \alpha$.

Remark 3.19. Thus, for any $\alpha \geq 0$, and any $r \in \mathbb{Q}$, we know that α^r exists. We can use this and the intermediate value theorem to define α^x for any real number x; in fact, this is the only direct way to construct the function $f(x) = \alpha^x$!

3.2.2 Extreme Value Theorem

The next important theorem from Calculus is the one which promises that on every closed interval, your continuous function must attain an absolute maximum and an absolute minimum. We can sketch simple examples of discontinuous functions, or of functions on intervals which are not closed or not bounded, to see that these hypotheses are all necessary.

Now what does it mean to say that the function attains its absolute max and absolute min? Well, that there exist x_1 and x_2 such that $f(x_1)$ is the minimum value and $f(x_2)$ is the maximum value; that is, for all x in our closed interval, we have

$$f(x_1) \leq f(x) \leq f(x_2).$$

Combining this with the Intermediate Value Theorem, we see that we can phrase the Extreme Value Theorem in the following simple way.

Theorem 3.20 (Extreme Value Theorem[2]). *Let f be a continuous function. If $I = [c, d]$ is any closed and bounded interval contained in the domain of f, then its direct image under f, that is,*

$$f(I) = \{f(x) \mid c \leq x \leq d\}$$

is also a closed and bounded interval, or else is a single point.

We will use Proposition 2.15, which says that when a convergent sequence is contained in a closed interval, its limit is, as well. Another key ingredient is the Bolzano-Weierstrass Theorem (Theorem 2.29).

Proof. Let $S = f(I) = \{f(x) \mid c \leq x \leq d\}$ be the image of the interval under f.

Let's show that S is bounded above, by contradiction. For suppose it is not. Then for each $n \in \mathbb{N}$, there is some $f(x_n) \in S$ such that $f(x_n) > n$. Now consider the sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[c, d]$. It could be divergent; but it is a bounded sequence, so by the Bolzano-Weierstrass theorem, it has a convergent subsequence, say $\{x_{n_k}\}_{k\in\mathbb{N}}$; let's write

$$\lim_{k\to\infty} x_{n_k} = x$$

and we have $x \in [c, d]$ as well. Since f is continuous, $\lim_{n\to\infty} f(x_{n_k}) = f(x)$. But this is impossible, since these terms are growing without bound: contradiction.

Thus S is bounded above; let $s = \sup(S)$.

With a similar argument we deduce S is bounded below; let $t = \inf(S)$.

We need to show that there exist $C, D \in [c, d]$ such that $f(C) = s$ and $f(D) = t$. By our STANDARD METHOD, construct a sequence of elements of S, call it $\{f(x_n)\}_{n\geq 1}$ which converges to s. Then the corresponding sequence $\{x_n\}_{n\geq 1}$ in $[c, d]$ need not be convergent; but by the Bolzano-Weierstrass theorem it has a convergent subsequence, say $\{x_{n_k}\}_{k\in\mathbb{N}}$, with limit $C \in [c, d]$. By continuity, we have

$$\lim_{n\to\infty} f(x_{n_k}) = f(C).$$

But since $\{f(x_n)\}_{n\geq 1}$ is convergent, any subsequence also converges to the same limit, namely s. Therefore $f(C) = s$.

A similar argument yields some $D \in [c, d]$ such that $f(D) = t$.

If $s = t$ then the function was constant. Otherwise, $s > t$ and by the Intermediate Value Theorem, we know that every y between $f(C)$ and $f(D)$ is equal to $f(z)$ for some z between C and D; that is, the entire interval $[f(D), f(C)]$ is contained in the image; and certainly the image is contained in this interval. Therefore $f(I) = [f(C), f(D)]$, which is what we wanted to show. □

Remark 3.21. The image of a closed and bounded interval under a continuous function is a closed and bounded interval. This fails for open and bounded intervals; for example, the function $f(x) = x^2$ sends the open interval $(-2, 2)$ to the non-open interval $[0, 4)$. It also fails for closed,

[2]Closed and bounded intervals are examples of compact and connected sets. The most general version of this theorem, which we do in MAT2125, is that the image of a compact set under a continuous function is compact, and the image of a connected set under a continuous function is connected.

unbounded intervals: consider $f(x) = 1/x$ and the interval $[5, \infty)$ which is sent to $(0, \frac{1}{5}]$. Therefore it is the combination of being closed and bounded (which we call compact) which is required for this theorem to hold.

3.2.3 Exercises

1. If you haven't done it before: prove that if $\{x_n\}_{n\geq 1}$ is a convergent sequence contained in (a, b) or in $[a, b]$ then its limit is contained in $[a, b]$.

2. Prove the intermediate value theorem for the two remaining cases: $f(x) = f(y)$ and $f(y) > f(x)$. Hint: the first is easy whereas the second is analogous to the proof given — easy once you understand that proof, but very hard if you don't.

3. Give an example of a (discontinuous) function which does not satisfy the conclusion of the intermediate value theorem.

4. Give an example of a (discontinuous) function which does not satisfy the conclusion of the extreme value theorem.

5. Use the intermediate value theorem to prove the existence of a root of the polynomial $p(x) = x^5 + x^4 + 3x^3 + 42x - 1$. Estimate a root to one decimal place.

6. Use the intermediate value theorem to prove that if f and g are continuous functions on an interval $[a, b]$ such that $f(a) < g(a)$ and $f(b) > g(b)$, then there exists a point $x \in [a, b]$ for which $f(x) = g(x)$.

7. Use the extreme value theorem to prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function which is a *contraction* (that is, for every $x, y \in \mathbb{R}$, $|f(x) - f(y)| < |x - y|$) then the image under f of any closed and bounded interval is a closed and bounded interval of strictly shorter length. (Note that the length of an interval $[a, b]$ is defined to be $b - a$.)

8. Use the intermediate value theorem to prove that if f and g are two continuous functions on an interval $[a, b] \subseteq [1, \infty)$ such tha t$g(x) \neq 0$ for all $x \in [a, b]$, and such that $0 < f(a)/a < g(a)/a$ and $f(b)/b > g(b)/b > 0$, then there is a point $x \in [a, b]$ such that $f(x) = g(x)$.

9. Let $f\colon [a, b] \to \mathbb{R}$ be a continuous function and let $\{x_n\}_{n\in\mathbb{N}}$ be a (possibly divergent) sequence in $[a, b]$. Prove that $\{f(x_n)\}_{n\in\mathbb{N}}$ has a convergent subsequence.

10. Let $f\colon [a, b] \to \mathbb{R}$ be a continuous, nonconstant function. Prove that there exists some $x \in [a, b]$ for which $f(x) \in \mathbb{Q}$.

11. Let $f\colon [a, b] \to \mathbb{R}$ be a continuous function such that for all $x \in [a, b]$, $f(x) \in \mathbb{Q}$. Prove that f is a constant function.

12. Give an example of a subset $A \subseteq \mathbb{R}$ and a continuous function $f\colon A \to \mathbb{R}$ such that there exist $a, b \in A$ and z between $f(a)$ and $f(b)$ such that there does NOT exist $x \in [a, b]$ for which $f(x) = z$. (Hint: compare this statement very carefully with the intermediate value theorem to spot the loophole.)

13. Prove there exists some $x \in \mathbb{R}$ for which $(x^2 - 3)e^{x^2+7}\sin(x) = 1$. (You may assume the continuity of all known continuous functions.)

Chapter 4

Derivatives and series

We studies derivatives extensively in MAT1320; our goal here is just to prove a key theorem (the Mean Value Theorem) and present the theory of Taylor polynomials and Taylor series.

4.1 Differentiable functions

4.1.1 Definition of a differentiable function

Definition 4.1. A function $f\colon (a,b) \to \mathbb{R}$ is called *differentiable* at a point $c \in (a,b)$ if

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

exists. In this case, the limit is denoted $f'(c)$ or $\frac{df}{dx}(c)$. We call a function simply differentiable if it is differentiable at every point in its domain.

In terms of sequences, this definition says that f is differentiable at c if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in (a,b) which converges to c, such that $x_n \neq c$ for all n, we have

$$\lim_{n\to\infty} \frac{f(x_n)-f(c)}{x_n-c} = f'(c).$$

Example 4.2. Let $f(x) = x^2$. Let's prove that f is differentiable at every $c \in \mathbb{R}$.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ which converges to c (but is never equal to it). We calculate

$$\begin{aligned}\lim_{n\to\infty} \frac{f(x_n)-f(c)}{x_n-c} &= \lim_{n\to\infty} \frac{x_n^2-c^2}{x_n-c}\\ &= \lim_{n\to\infty}(x_n+c)\\ &= c+c = 2c\end{aligned}$$

by the algebra of convergent sequences. We get the same answer for all sequences $\{x_n\}_{n\in\mathbb{N}}$ so f is differentiable at c and $f'(c) = 2c$.

Example 4.3. Let $f(x) = |x|$. Show that f is not differentiable at 0.

Set $x_n = \frac{1}{n}$; this is a sequence in the domain of f which never equals 0 and which converges to 0. Since $f(x_n) > 0$ for all n, we have

$$\lim_{n\to\infty} \frac{f(x_n) - f(c)}{x_n - c} = \lim_{n\to\infty} \frac{x_n - 0}{x_n - 0} = 1.$$

On the other hand, the sequence $x_n = -\frac{1}{n}$ is another nonzero sequence in the domain of f which converges to 0, but since $f(x_n) < 0$ for all n, we have instead

$$\lim_{n\to\infty} \frac{f(x_n) - f(c)}{x_n - c} = \lim_{n\to\infty} \frac{-x_n - 0}{x_n - 0} = -1.$$

Thus the limit of the quotient $\frac{|x|-0}{x-0}$ does not exist as x tends to 0; whence the function is not differentiable at 0.

Remark 4.4. It is true that, for $f(x) = |x|$,

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

(see the exercises). That said, we just saw that we cannot deduce the value of $f'(0)$ from this same formula. Even worse: you might want to conclude that $f'(0)$ does not exist simply because there is no possible value of $f'(0)$ which would make f' a continuous function — but there exist functions which are differentiable everywhere yet their derivatives are not continuous! (see the exercises).

4.1.2 Consequences of the definition of differentiability

Proposition 4.5.

1. *Let f and g be functions differentiable at c and $r \in \mathbb{R}$. Then rf, $f+g$ and fg are continuous at c. Moreover, $(rf)'(c) = rf'(c)$, $(f+g)'(c) = f'(c)+g'(c)$ and $(fg)'(c) = f'(c)g(c)+f(c)g'(c)$.*

2. *If additionally $g(c) \neq 0$ then $\frac{1}{g}$ is differentiable at c and the quotient rule holds:*

$$\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{(g(c))^2}.$$

3. *If the composition $f \circ g$ is defined at c and g is differentiable at c, and f is differentiable at $g(c)$, then $f \circ g$ is differentiable at c. Furthermore, $(f \circ g)'(c) = f'(g(c))g'(c)$.*

4. *The following functions are differentiable on their domains: e^x, $\sin(x)$, $\cos(x)$, $\ln(x)$, $\arctan(x)$.*

5. *For each $r \in \mathbb{R}_{>0}$, the function x^r is differentiable on $(0, \infty)$.*

6. *The functions $\arcsin(x)$ and $\arccos(x)$ are differentiable on the open interval $(-1, 1)$.*

We will assume this result for the rest of the course; you will see (or have seen) the proofs at various points over your mathematical career.

Here are some more interesting properties of differentiable functions, proven using the definition of differentiability and the derivative.

Proposition 4.6. *If f is differentiable at c then it is continuous at c.*

Proof. Suppose that f is differentiable at c. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in the domain of f converging to c and distinct from c. Then

$$\begin{aligned}\lim_{n\to\infty}(f(x_n)-f(c)) &= \lim_{n\to\infty}\left(\frac{f(x_n)-f(c)}{x_n-c}(x_n-c)\right)\\ &= \lim_{n\to\infty}\frac{f(x_n)-f(c)}{x_n-c}\lim_{n\to\infty}(x_n-c)\\ &= f'(c)\cdot 0\\ &= 0\end{aligned}$$

which implies that $\lim_{n\to\infty} f(x_n) = f(c)$, as required. □

Lemma 4.7. *Suppose $f\colon (a,b)\to\mathbb{R}$ is a differentiable function and $c\in(a,b)$ is a relative extremum. Then $f'(c)=0$.*

Proof. Let's show that if f attains a local maximum at c then $f'(c)=0$; the case of a local minimum is an exercise.

Since f has a local maximum at c, there is some interval $(c-\delta, c+\delta)$ about c such that for every $x\in(c-\delta,c+\delta)$, such that $x\neq c$, we have $f(x)<f(c)$.

So let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence contained in $(c-\delta,c)$ which converges to c. For every $n\in\mathbb{N}$ we have

$$\frac{f(x_n)-f(c)}{x_n-c}>0$$

so we conclude that in the limit we obtain $f'(c)\geq 0$.

On the other hand, let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence contained in $(c,c+\delta)$ which converges to c. For every $n\in\mathbb{N}$ we have

$$\frac{f(x_n)-f(c)}{x_n-c}<0$$

so we conclude that in the limit we have $f'(c)\leq 0$.

Since f is differentiable, both statements are true, so $f'(c)=0$. □

4.1.3 The Mean Value Theorem

The Mean Value Theorem says: the average rate of change of a function on an interval is equal to the instantaneous rate of change of the function at some point in that interval. It is intuitively clear: if my average velocity on a trip is 80 km/h, then surely my spedometer must have read 80 km/h at some instant during my trip!

We begin with a simple case of the mean value theorem, when the endpoints of the interval are on the x-axis, so that the average rate of change is just 0.

Theorem 4.8 (Rolle's theorem)**.** *Suppose f is a continuous function which is defined on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then there is some $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. By the Extreme Value Theorem, there are numbers C, D with $a \leq C, D \leq b$ such that for all $x \in [a, b]$, $f(C) \leq f(x) \leq f(D)$. If $f(C) = f(D)$ then f is a constant function, which has derivative 0 everywhere, from the definition.

Otherwise, $f(C) < f(D)$ and so, since $f(a) = f(b)$ at least one of C or D must be different from a or b, that is, it is contained in the open interval (a, b). But then this is a local extremum, and so the derivative at that point is 0, by the lemma. □

The full mean value theorem (théorème des accroissements finis) was proven by Lagrange in 1787. We used it in MAT1320 to prove the Fundamental Theorem of Calculus.

Theorem 4.9 (Mean Value Theorem)**.** *Suppose f is a continuous function, defined on $[a, b]$, and differentiable on (a, b). Then there is some $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a).$$

Proof. The main idea in this proof is to "straighten out" our function f so that we can apply Rolle's theorem. By looking at a graph, we see that we want to substract from $f(x)$ the linear function passing through the point $(a, f(a))$ and the point $(b, f(b))$; of course this line has slope

$$m = \frac{f(b) - f(a)}{b - a}$$

so therefore the equation of the line is $y - f(a) = m(x - a)$ since we pass through $(a, f(a))$. This gives $y = L(x)$ where

$$L(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

Ok, so let's try that.

Define a new function h by

$$h(x) = f(x) - L(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

It is clearly continuous, defined on $[a, b]$ and differentiable on (a, b). Moreover, we see that

$$h(a) = f(a) - f(a) - 0 = 0, \quad h(b) = f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a) = 0$$

so by Rolle's theorem, there is some $c \in (a, b)$ such that $h'(c) = 0$.

We have

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

so if $h'(c) = 0$ then

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

as required. □

We can also prove a stronger version of the Mean Value Theorem, which is due to Cauchy in 1821 (34 years after Lagrange proved the original); see the exercises.

4.1.4 Exercises

1. Prove that the function $f(x) = x$ is continuous. Deduce that all polynomial functions are continuous, using the theorem about the properties of continuous functions.

2. Prove the sum, product and quotient rules. (For the quotient rule: first prove that if g is a continuous function such that $g(c) \neq 0$, then there exists an interval $(c - r, c + r)$ on which $g(x) \neq 0$.)

3. (challenging) Prove the chain rule.

4. Suppose g is a bounded but not necessarily continuous or differentiable function. Prove that $x^2 g(x)$ is differentiable at 0. Give an example to show that $xg(x)$ is not necessarily differentiable at 0.

5. Are the following functions differentiable at c?

   (a) $f(x) = \begin{cases} x^2 & \text{if } x \geq 2 \\ 6 - x & \text{if } x < 2 \end{cases}, \quad c = 2$

   (b) $f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}, \quad c = 0$

6. Let $f \colon \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} 1 & \text{if } x \leq -1 \\ x^2 & \text{if } -1 < x < 1 \\ 1 & \text{if } x \geq 1. \end{cases}$$

   Show that f is continuous on $[-1, 1]$ but that it is not differentiable at either $c = -1$ or $c = 1$. May we apply Rolle's theorem to f on the interval $[-1, 1]$?

7. Let f be a function defined in multiple parts. Let g be a function which is differentiable on its domain.

   (a) Show that if U is an open interval such that $f(x) = g(x)$ for each $x \in U$, then f is differentiable at every $x \in U$.

(b) Give an example of a closed interval U, and suitable functions f and g meeting the above hypotheses, such that $f(x) = g(x)$ for every $x \in U$ but such that f is not differentiable at an endpoint of U.

8. Let g and h be differentiable functions such that g' and h' are continuous. Let f be defined by

$$f(x) = \begin{cases} g(x) & \text{if } x < c \\ h(x) & \text{if } x \geq c. \end{cases}$$

Show that if $g(c) = h(c)$ and $g'(c) = h'(c)$ then f is differentiable at c, hence at every x.

9. Suppose that f is a function whose derivative is continuous. Give another proof of the mean value theorem, this time using the intermediate value theorem applied to f'.

10. Prove that the following function is continuous, and differentiable everywhere, but that its derivative is not continuous. (This explains why we couldn't use the Intermediate Value Theorem on f' to prove the Mean Value Theorem for f, in general.)

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Hint: since $x^2 \sin(1/x)$ is a composition of differentiable functions for $x \neq 0$, it is differentiable. Therefore there are just two things to check: is it differentiable at $c = 0$ and is it true that the derivative is continuous.

11. Cauchy (1789–1857) proved a generalization of the Mean Value Theorem, which we state as follows:

Suppose f and g are continuous functions, defined on $[a, b]$ and differentiable on (a, b). Suppose $g(b) \neq g(a)$. Then there exists some $c \in (a, b)$ such that $g'(c) \neq 0$ and

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

(a) Explain how the Mean Value Theorem (as stated in class) is a special case of Cauchy's generalization.

(b) Define

$$h(x) = f(x) - f(a) - (g(x) - g(a)) \left(\frac{f(b) - f(a)}{g(b) - g(a)} \right).$$

Justify that h satisfies the hypotheses of Rolle's theorem and use this to deduce a proof of Cauchy's Mean Value Theorem.

(c) (bonus) Prove the slightly more general version (with fewer hypotheses on g: Suppose f and g are continuous functions, defined on $[a, b]$ and differentiable on (a, b). Then there exists some $c \in (a, b)$ such that $f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a))$.

12. L'Hospital's rule (or rather, one case of it) can be stated as:

Suppose f and g are differentiable on $\mathbb{R}$ and $f(a) = g(a) = 0$. If

$$\lim_{x \to a} \frac{f'(x)}{g'(x)}$$

exists and is equal to $\lambda \in \mathbb{R}$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lambda$$

as well. Your goal in this question is to prove this using Cauchy's Mean Value Theorem. Please assume for simplicity that $g(x) \neq 0$ for any $x \neq a$.

The question: show, under the given hypotheses, that for any sequence $\{x_n\}_{n \geq 1}$ converging to a, such that $x_n \neq a$ for any n, we have $\lim_{n\to\infty}(f(x_n)/g(x_n)) = \lambda$.

4.2 Applications

4.2.1 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus states, roughly, that differentiation and integration are opposite operations (which justifies our use of the indefinite integral as the notation for an anti-derivative). To prove it, we first prove a lemma that uses the Mean Value Theorem.

Recall how we defined a Riemann sums of f on $[a, b]$. Let $n \in \mathbb{N}_+$. Set $\Delta x = (b - a)/n$ and for each $i \in \{0, 1, 2, \cdots, n\}$, set $x_i = a + i\Delta x$. Then a Riemann sum of f on $[a, b]$ is any sum of the form

$$\sum_{i=1}^{n} f(\xi_i)\Delta x$$

for some choice of ξ_i in the interval $[x_{i-1}, x_i]$. These Riemann sums represent the (signed) sum of the areas of rectangles which try to match the area under the curve (i.e. the integral). No matter how the ξ_i are chosen, if f is integrable[1] then the sums converge to $\int_a^b f(x)\, dx$ as the number of intervals n goes to infinity.

Lemma 4.10. *Suppose $F\colon [a, b] \to \mathbb{R}$ is a continuous function, differentiable on (a, b), such that $F'(x) = f(x)$ for all $x \in (a, b)$. Then for any $n \in \mathbb{N}$, there is a Riemann sum S_n of f on $[a, b]$ with n parts such that $S_n = F(b) - F(a)$.*

Proof. The hypotheses of the Mean Value Theorem apply to F on each interval $[x_{i-1}, x_i]$. Therefore there exists $\xi_i \in (x_{i-1}, x_i)$ such that

$$F(x_i) - F(x_{i-1}) = f(\xi_i)(x_i - x_{i-1}) = f(\xi_i)\Delta x$$

Taking the sum over all i from 1 to n of the right hand side yields a Riemann sum S_n with n parts; taking the sum over all i from 1 to n of the left hand side yields

$$(F(x_n) - F(x_{n-1})) + F(x_{n-1}) - F(x_{n-2}) + \cdots + (F(x_1) - F(x_0)) = F(x_n) - F(x_0) = F(b) - F(a).$$

□

[1] We define integrability carefully in MAT2125. For now: any continuous function is integrable.

Theorem 4.11 (The Fundamental Theorem of Calculus, Part II)**.** *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Suppose $F\colon [a,b] \to \mathbb{R}$ is a continuous function which is differentiable on (a,b) and $F'(x) = f(x)$ for all $x \in (a,b)$. Then*

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Proof. Since f is continuous, its integral $\int_a^b f(x)\, dx$ exists and is the limit of any sequence of Riemann sums. By the lemma, there exists a sequence of Riemann sums $\{S_n\}_{n\in\mathbb{N}_+}$ such that $S_n = F(b) - F(a)$ for all n. Thus it is a constant sequence, with limit $F(b) - F(a)$; since it is a limit of Riemann sums, it is equal to the integral. □

Remark 4.12. Part I of the theorem is the other implication, that f having an integral implies it has an anti-derivative; one can prove it using the *Mean Value Theorem for integrals* (MAT2125). We state it here for completeness, but will not prove it.

Theorem 4.13 (The Fundamental Theorem of Calculus, Part I)**.** *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Define a function $F\colon [a,b] \to \mathbb{R}$ by*

$$F(x) = \int_a^x f(t)\, dt.$$

Then F is differentiable on (a,b) with derivative f.

4.2.2 Increasing and decreasing functions

The mean value theorem has many applications, because it is exactly the tool you need when you want to say something about how f behaves on an interval based on how it behaves instantaneously at every point. Here is a typical example.

Lemma 4.14. *Suppose $f\colon (a,b) \to \mathbb{R}$ is differentiable. Then if for all $x \in (a,b)$, we have*

- $f'(x) > 0$*, then f is increasing*
- $f'(x) < 0$*, then f is decreasing*
- $f'(x) = 0$*, then f is constant.*

This seems impossible to prove: how can you convert instantaneous information to something global? If we stare at the definition we see it is close to hopeless. But it's not the definition we'll use, it's the Mean Value Theorem.

Proof. Let's prove the first case; the rest are an exercise.

Assume that $f'(x) > 0$. To prove that f is increasing, we need to prove that for every $x < y$ in the interval (a,b), we have $f(x) < f(y)$.

So let x, y be such that $a < x < y < b$. By the Mean Value Theorem, there is some $c \in (x, y)$ such that

$$f(y) - f(x) = f'(c)(y - x).$$

Since $c \in (x, y) \subseteq (a, b)$, we know $f'(c) > 0$ by hypothesis. Also $y > x$ so $y - x > 0$. Therefore $f(y) - f(x) > 0$, as required. □

4.2.3 More uses of derivatives: Taylor approximations

Now that we've proven the three main theorems that form the fundaments of Calculus of one variable (that is, the Intermediate Value Theorem, which tells us that continuous functions can't skip values; the Extreme Value Theorem, which tells us that every continuous function on a closed and bounded interval does attain its maximum and minimum values; and the Mean Value Theorem, which lets us use information about the derivative to find out about the function), let's push onwards with the development of Calculus, and explore a couple of topics in differentiation and integration of functions of a single variable.

See [S, Sections 8.5 to 8.7]; we are covering the highlights.

We used the *linear approximation* of f at a in Calculus I, as a crude way of estimating the value of f near a. The linear approximation is just the tangent line, viewed as a function:

$$L(x) = f(a) + f'(a)(x - a).$$

But what if we wanted to approximate f by a polynomial function near a? Couldn't that make a better fit?

Indeed, suppose we have

$$p(x) = b_0 + b_1(x - a) + b_2(x - a)^2 + b_3(x - a)^3 + \cdots + b_n(x - a)^n$$

(a polynomial written so it's "centered at $x = a$"). Then

$$\begin{aligned} p(a) &= b_0 \\ p'(a) &= b_1 \\ p''(a) &= 2b_2 \\ p'''(a) &= 6b_3 \end{aligned}$$

and in general, $p^{(n)}(a) = n!b_n$. So the coefficients are telling us about the derivatives of p at a; hence if we want to find $p(x)$ which best matches $f(x)$, we should just match derivatives at a; our "best guess polynomial" of degree n is therefore

$$T_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

called the *Taylor polynomial of degree n of f centered at a*. We sometimes, as needed, add f and a to the notation ($T^f_{n,a}(x)$), if there is more than one function or point in question.

Example 4.15. Let $f(x) = e^x$ and choose $a = 0$. Then $f^{(n)}(x) = e^x$ for all n, which gives $f^{(n)}(0) = e^0 = 1$, so we have

$$T_n(x) = 1 + x + \frac{1}{2}x^2 + \cdots + \frac{1}{n!}x^n$$

In a similar way, we derive the Taylor polynomial of e^x at any point a, for any degree n:

$$T_{n,a}(x) = e^a + e^a(x-a) + \frac{e^a}{2}(x-a)^2 + \cdots + \frac{e^a}{n!}(x-a)^n.$$

Example 4.16. Let $f(x) = \ln(x)$ and choose $a = 1$. Then we have

	$f^{(n)}(x)$	$f^{(n)}(1)$	$\frac{f^{(n)}(1)}{n!}(x-1)^n$
$f(x)$	$\ln(x)$	0	0
$f'(x)$	$\frac{1}{x} = x^{-1}$	1	$(x-1)$
$f''(x)$	$-x^{-2}$	-1	$-\frac{1}{2}(x-1)^2$
$f'''(x)$	$2x^{-3}$	2	$\frac{2}{3!}(x-1)^3$
$f^{(4)}(x)$	$-6x^{-4}$	$-(3!)$	$\frac{-3!}{4!}(x-1)^4$

So the Taylor polynomial of degree 4 of ln at 1 is

$$T_4(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$$

Example 4.17. Let's calculate the Taylor polynomials of the function $f(x) = \cos(x)$, centered at 0.

	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}x^n$
$f(x)$	$\cos(x)$	1	1
$f'(x)$	$-\sin(x)$	0	0
$f''(x)$	$-\cos(x)$	-1	$-\frac{1}{2}x^2$
$f'''(x)$	$\sin(x)$	0	0
$f^{(4)}(x)$	$\cos(x)$	1	$\frac{1}{4!}x^4$

Thus

$$T_0(x) = T_1(x) = 1; T_2(x) = T_3(x) = 1 - \frac{1}{2}x^2$$

and

$$T_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4.$$

We can easily deduce (from the cyclical pattern of the derivative) that

$$T_n(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + (-1)^n\frac{1}{(2n)!}x^{2n}.$$

The first few approximations as sketched versus the graph of $y = \cos(x)$ in Fig.4.1.

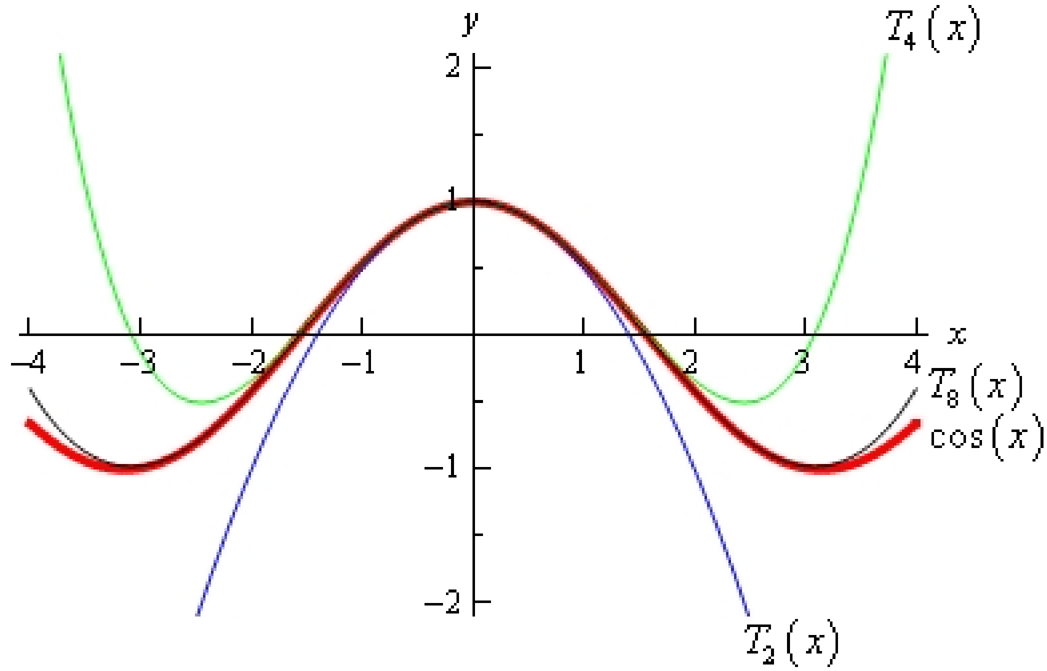


Figure 4.1: The function $\cos(x)$ compared with its first few Taylor polynomials. This image comes from the website `http://tutorial.math.lamar.edu/Classes/CalcII/TaylorSeriesApps.aspx` of which author Paul Dawkins retains all copyright.

4.2.4 Taylor's theorem

OK, so these approximations aren't too hard to write down; how good are they? Is this a worthwhile thing to do?

Theorem 4.18 (Taylor's theorem)**.** *Suppose f is $n+1$-times differentiable in a. Then $T_n(x)$ exists and we can write*

$$f(x) = T_n(x) + R_n(x)$$

so that $R_n(x)$ is the remainder (error) of the Taylor approximation. Then for each x in the domain of f, there is a number c between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

(Note that this is similar, but not equal, to $T_{n+1}(x) - T_n(x)$.)

Now the c that shows up in the remainder term varies for each x, so a more practical way of interpreting this last condition is as follows.

Suppose $d > 0$. If $|f^{(n+1)}(x)| \leq M$ for all x with $|x-a| \leq d$ then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

for all x with $|x-a| \leq d$.

We will leave the proof, which uses a generalization of the Mean Value Theorem, as an interesting exercise.

Example 4.19. Consider $f(x) = e^x$. For any $d > 0$, we have that on $[-d, d]$, $|f^{(n+1)}(x)| \leq e^d$ so let $M = e^d$. Then the error satisfies

$$|R_n(x)| \leq \frac{e^d}{(n+1)!}|x|^{n+1}$$

for all $x \in [-d, d]$. For any fixed d, this error goes to 0 as $n \to \infty$ (exercise).

So in particular: yes, on any fixed interval $[-d, d]$, you can make $|f(x) - T_n(f)(x)| < \varepsilon$ (whatever $\varepsilon > 0$ you want) by choosing n large enough.

Example 4.20. Consider $f(x) = \ln(x)$ centered at $a = 1$, on an interval $[\frac{1}{2}, \frac{3}{2}]$. Now

$$f^{(n+1)}(x) = (-1)^n n! x^{-(n+1)}$$

so if $x \in [\frac{1}{2}, \frac{3}{2}]$ we have

$$|f^{(n+1)}(x)| \leq \frac{n!}{(\frac{1}{2})^{n+1}} = n! 2^{n+1}$$

which is huge; but when we calculate

$$\begin{aligned} |R_n(x)| &\leq \frac{2^{n+1}}{(n+1)}|x-1|^{n+1} \\ &\leq \frac{2^{n+1}}{n+1}\left|\frac{1}{2}\right|^{n+1} \\ &= \frac{1}{n+1} \end{aligned}$$

we see that the error tends to 0 as $n \to \infty$.

Hence the Taylor polynomial can be made as good an approximation as you like on the interval $[\frac{1}{2}, \frac{3}{2}]$, by choosing n large enough.

Example 4.21. For each $n \in \mathbb{N}$, find an upper bound on the Taylor remainder $R_n(x)$ of the funciton $\cos(x)$ centered at 0, for values of x in the interval $[r, r]$.

Solution: Since $f^{(n)}(x)$ is one of the 4 functions $\pm\sin(x)$ or $\pm\cos(x)$, it follows that for all $n \in \mathbb{N}$ and for all $x \in \mathbb{R}$, we have $|f^{(n+1)}(x)| \leq 1$. Thus for all $x \in [-r, r]$, we have

$$|R_n(x)| = \frac{|f^{(n+1)}(x)|}{(n+1)!}|x-0|^{n+1} \leq \frac{|x|^{n+1}}{(n+1)!} \leq \frac{r^{n+1}}{(n+1)!}.$$

For example, at $x = \pi$, we have $\cos(1) \simeq 0.54030$ whereas $T_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 \simeq 0.54167$; that's not bad. On the interval $[-1, 1]$ we had estimated that $|R_4(x)| \leq \frac{1}{5!} \simeq 0.00833$ whereas in fact $f(1) - T_4(1) = -0.00137$. So our upper bound on the error term was correct. (If our estimate had come out lower than the actual error, then we would have known we had made a mistake.)

On the other hand, note that on $[-\pi, \pi]$, our upper bound on $|R_n(x)|$ is $\pi^5/5! \simeq 2.55$; this isn't very good, but we can see why it comes out so high in looking at Fig. 4.1.

Nevertheless, note that for each constant r, $\lim_{n\to\infty} \frac{r^{n+1}}{(n+1)!} = 0$, whence if we fix an interval $[-r, r]$ then for every x in that interval we have

$$\lim_{n\to\infty} R_n(x) = 0.$$

This means that for each x we can choose n sufficiently large to ensure that $T_n(x)$ is a good approximation of $\cos(x)$.

Remark 4.22. More generally the idea is: fix $d > 0$ and find out the maximum value that $|f^{(n+1)}(x)|$ takes on $[a-d, a+d]$. Call it M. Then you have a bound on the error of the Taylor polynomial approximation, which for large n you expect will be quite small.

In the above cases, we found a much stronger result, that $\lim_{n\to\infty} R_n(x) = 0$, so as long as we pick n large enough, $T_n f$ is as good an approximation as we like to f. Let's push this idea to its natural limit.

But first, we need to decide what an "infinite sum" means — and what can go wrong.

4.2.5 Exercises

1. Prove the rest of Lemma 4.14.
2. Prove that if $f'(x) = 0$ on an interval then f is constant on that interval.
3. Show that any polynomial f of degree n is equal to its nth Taylor polynomial. What is $T_{n+1}(f)$?
4. As functions, $f'(x) = \frac{df}{dx}(x)$. So if a is a constant, then $f'(a)$ means: take the derivative and then evaluate it at a. What is the difference between $f'(a)$ and $\frac{d}{dx}(f(a))$? Or more specifically, what is the difference between $f'(2)$ and $\frac{d}{dx}(f(2))$?
5. Use Cauchy's Mean Value Theorem to prove Taylor's theorem. That is, suppose f is an $(n+1)$-times differentiable function on a domain (b, d) and $a \in (b, d)$. Let $T_n(f)$ be the Taylor polynomial of degree n centered at a of f. Show that for each $x \in (b, d)$ there is some c between a and x such that
$$f(x) - T_n(f)(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}.$$
Steps: Fix $x > a$ for your proof; the case $x < a$ is similar and you don't need to discuss it. Define
$$S(t) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k$$
Verify that the hypotheses of Cauchy's Mean Value Theorem are satisfied by the functions $S(t)$ and $g(t) = (x-t)^{n+1}$. Carefully compute $S'(t)$; it will simplify beautifully. Apply the theorem and deduce Taylor's theorem from the result.
6. For each of the following functions: compute $T_3(x)$, centered at a; explicitly compute $f(z) - T_3(z)$ for the z given; determine an upper bound for $f^{(4)}(x)$ for all $x \in [-z, z]$; deduce an upper bound on $R_3(z)$ using Taylor's theorem; and compare this bound with the exact value you computed earlier.

(a) $f(x) = \sin(x)$, $a = 0$, $z = \pi/2$

(b) $f(x) = \tan(x)$, $a = 0$, $z = 0.1$

(c) $f(x) = \sqrt{1+x}$, $a = 1$, $z = \frac{1}{2}$

7. (a) Explicitly compute the degree 4 Taylor polynomial for $\sec(x)$ at $a = 0$.

   (b) We know that $\cos(x)\sec(x) = 1$; in this question we ask what happens when we multiply the Taylor polynomials. Compute $h(x)$, the product of the two functions $T_4(\sec)$ and $T_4(\cos)$. Verify that $h(x) - 1$ has no terms of degree less than 5.

4.3 Series

4.3.1 Definition of series

A special kind of sequence is one where each term is given by adding a number to the previous term. We call this a *series* and define it as follows.

Definition 4.23. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers, and for each $n \geq 0$ define

$$s_n = \sum_{k=0}^{n} a_k.$$

Then the sequence $\{s_n\}_{n\geq 0}$ is called a *series* and each s_n is called a *partial sum*; the *terms* of the series are the numbers a_n that we add together.

This notation is quite cumbersome, so instead we write "the series $\displaystyle\sum_{k=0}^{\infty} a_k$" or "the series $\displaystyle\sum_{n\geq 1} a_n$" to mean this sequence of partial sums.

Definition 4.24. The series $\sum_{k=0}^{\infty} a_k$ *converges* if the sequence $\{s_n\}_{n\geq 0}$ of partial sums converges. If $\lim_{n\to\infty} s_n = s$ then we write

$$\sum_{k=0}^{\infty} a_k = s$$

and call s the *sum* of the series. If $\{s_n\}_{n\geq 0}$ diverges, then we say $\sum_{k=0}^{\infty} a_k$ diverges.

Note that, like sequences, series may start at different values, as in the following examples; however, unlike sequences, the starting point is very important (because starting at a different point will change every element of the sequence of partial sums, and hence change the limit).

4.3.2 Examples

Example 4.25. Let r be a fixed real number. The series

$$\sum_{k=0}^{\infty} r^k$$

is called a *geometric series*. The nth partial sum is of this series is

$$\sum_{k=0}^{n} r^k = 1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

If $-1 < r < 1$, then $\lim_{n\to\infty} r^{n+1} = 0$ so by the algebra of convergent sequences, we see that the sequence of partial sums, hence the series, converges to limit $1/(1-r)$. Thus

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \quad \text{if } -1 < r < 1.$$

If $r = 1$, or if $|r| > 1$, then the sequence of partial sums is not bounded hence it is divergent, so the series diverges.

If $r = -1$, then note that

$$\sum_{k=0}^{n} (-1)^n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

and so the series is also divergent for $r = -1$.

Example 4.26. Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}.$$

So the terms of this series are

$$a_k = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Thus the partial sums are

$$\begin{aligned} \sum_{k=1}^{n} \frac{1}{k(k+1)} &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

and the limit of this sequence of partial sums is 1. Therefore the series converges and we have

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1.$$

Example 4.27. The *harmonic series* is the series

$$\sum_{k=1}^{\infty} \frac{1}{k}.$$

Let's show that it *diverges*.

Method 1: The nth partial sum is

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

so the 2^nth partial sum is

$$s_{2^n} = \underbrace{1 + \frac{1}{2}}_{>\frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>\frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{>\frac{1}{2}} + \cdots + \underbrace{\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}}_{>\frac{1}{2}}$$
$$> \frac{1}{2}n.$$

So this subsequence of partial sums (to the 2^nth term) diverges, whence the series diverges.

Method 2: Each partial sum of this series over-estimates de area under the curve $y = 1/x$ between $x = 1$ and $x = n + 1$. Since this is

$$\int_1^{n+1} \frac{1}{x}\, dx = \ln(n+1) - \ln(1) = \ln(n+1)$$

which diverges to ∞, it follows that the partial sums diverge to ∞. So the series diverges, albeit very slowly.

We write

$$\sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

Proposition 4.28. *If* $\sum_{n\geq 0} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

Equivalently (and more usefully) this says:

If $\lim_{n\to\infty} a_n \neq 0$, or if $\{a_n\}_{n\in\mathbb{N}}$ diverges, then the series $\sum_{n\geq 0} a_n$ diverges.

Remark 4.29. Always remember the harmonic series: it is divergent yet its terms tend to zero.

4.4 Convergence tests for series with positive terms

Now let us restrict out attention to series $\sum_{n=0}^{\infty} a_n$ where $a_n > 0$ for all $n \in \mathbb{N}$.

4.4.1 Comparison test

If the series has positive terms, then its partial sums satisfy

$$s_{n+1} = a_{n+1} + s_n > s_n$$

so that $\{s_n\}_{n\geq 0}$ is *increasing*. Thus from Theorem 2.25 we deduce:

A series with positive terms is convergent iff its partial sums are bounded above.

This doesn't seem so useful at first — after all, if you keep adding terms how can you figure out if your series is like the harmonic series, that never stops growing, or like the geometric series, which is bounded — until you realize that the trick is to compare one series to another.

Theorem 4.30 (Comparison Theorem)**.** *Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be sequences of real numbers such that for all $n \in \mathbb{N}$, we have*[2]

$$0 \leq a_n \leq b_n.$$

Then

1. *If $\sum_{n\in\mathbb{N}} b_n$ converges then $\sum_{n\in\mathbb{N}} a_n$ converges.*
2. *If $\sum_{n\in\mathbb{N}} a_n$ diverges then $\sum_{n\in\mathbb{N}} b_n$ diverges.*

Proof. Let us first suppose that $\sum_{n\in\mathbb{N}} b_n$ converges to b. The sequence of partial sums is increasing, so every partial sum is less than or equal to b. Then the given inequalities give, for every $n \in \mathbb{N}$:

$$\sum_{k=0}^{n} a_n \leq \sum_{k=0}^{n} b_n \leq b.$$

Thus the partial sums of $\sum_{n\in\mathbb{N}} a_n$, which form an increasing sequence, are bounded above by b, so $\sum_{n\in\mathbb{N}} a_n$ converges (to some value between 0 and b).

Now suppose instead that $\sum_{n\in\mathbb{N}} a_n$ diverges. Since the sequence of partial sums is increasing, we conclude that in fact

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_k = \infty.$$

Thus for every $K \in \mathbb{N}$ there is some $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$K \leq \sum_{k=0}^{n} a_k$$

and this this is less than or equal to $\sum_{k=0}^{n} b_k$ we conclude that

$$\lim_{n\to\infty} \sum_{k=0}^{n} b_k = \infty$$

as well. Hence the series $\sum_{n\in\mathbb{N}} b_n$ diverges. □

[2] It suffices, in fact, that there is some $N \in \mathbb{N}$ such that for all $n \geq N$ we have $0 \leq a_n \leq b_n$, because a finite number of terms are completely irrelevant when it comes to talking about convergence — it's only what happens as $n \to \infty$ that matters.

Example 4.31. Consider $\sum_{n\geq 1} \frac{1}{(n+1)^2}$. We have

$$\frac{1}{(n+1)^2} \leq \frac{1}{n(n+1)}$$

and since $\sum_{n\geq 1} \frac{1}{n(n+1)}$ converges, by the comparison test, we know $\sum_{n\geq 1} \frac{1}{(n+1)^2}$ converges.

Note that

$$\sum_{n\geq 1} \frac{1}{(n+1)^2} = \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

Since this converges, so does

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{n\geq 1} \frac{1}{n^2}$$

(though it has a different sum!). In fact, the sum of this latter series is $\frac{\pi^2}{6}$, as proven by Euler[3].

Example 4.32. We can use the comparison test to show that

$$\sum_{n\geq 1} \frac{1}{n^k}$$

converges for any $k \geq 2$. (Exercise)

Remark 4.33. In fact, it can be shown that

$$\sum_{n\geq 1} \frac{1}{n^k}$$

converges for $k > 1$ and diverges for $k \leq 1$.

The comparison test can be used to prove the convergence or divergence of series whose terms are rational functions with positive values. It can require some ingenuity to find the right comparison, however!

Example 4.34. Determine the convergence or divergence of the series $\sum_{n\geq 1} a_n$ where $a_n = \frac{3n^2+2n-1}{5n^3+2}$.

Solution: this rational function has total degree -1, like $\frac{1}{n}$, so we expect it to diverge — the terms simply won't tend to zero fast enough for this series to converge. Therefore we are seeking to find $\alpha \in \mathbb{R}$ and $k \in \mathbb{Z}$ (nicest choice is $k = 0$, of course) so that

$$\frac{3n^2 + 2n - 1}{5n^3 + 2} \geq \frac{\alpha}{n + k}.$$

We proceed carefully: for $n \geq 1$ we have $2n - 1 \geq 0$ and $2 \leq 2n^3$ so:

$$\frac{3n^2 + 2n - 1}{5n^3 + 2} \geq \frac{3n^2}{5n^3 + 2n^3} = \frac{3/7}{n}.$$

[3] For an overview of how it's done: `http://www.math.uu.se/~bjorklund/euler.pdf`.

Now if $\displaystyle\sum_{n\geq 1} \frac{3/7}{n}$ were convergent, then by the algebra of convergent sequences, we could multiply it by 7/3 and conclude that the harmonic series converges. This is a contradiction; hence $\displaystyle\sum_{n\geq 1} \frac{3/7}{n}$ (and indeed, any scalar multiple of the harmonic series) diverges.

Therefore by the comparison test, $\sum_{n\geq 1} a_n$ diverges as well.

Note that there are zillions of different comparisons that you could make — there's nothing special about the choices made in the preceding example EXCEPT that you should never change the degree of your rational function. If you start off with something whose net degree is -1 and in the course of making comparisons end up with something with net degree -2 then it's time to scrap your work and start over, because obviously that will never lead you to a good answer.

Example 4.35. Consider

$$\sum_{n=2}^{\infty} \frac{4n^2 - 7n + 20}{n^4 - 1}$$

We see that the total degree is -2 so it is comparable to the series $\sum \frac{1}{n^2}$; therefore we expect it to converge.

We thus make comparisons as follows (for example; you may have another sequence of steps that achieves the same basic result we need). Note that since we want to prove it will converge, we have to show it is LESS THAN something like $\frac{1}{n^2}$ (see the comparison test).

$$\begin{aligned}
\frac{4n^2 - 7n + 20}{n^4 - 1} &< \frac{4n^2 + 20}{n^4 - 1} && \text{since } -7n < 0 \\
&< \frac{4n^2 + 20n^2}{n^4 - 1} && \text{since } 20 < 20n^2 \text{ for } n \geq 2 \\
&< \frac{24n^2}{n^4 - n^3} && \text{since } -1 > -n^3 \text{ so denom. smaller} \\
&= \frac{24}{n(n-1)} && \text{simplifying} \\
&< \frac{24}{(n-1)^2} && \text{since } n > n-1 \text{ so denom. smaller}
\end{aligned}$$

Finally, note that

$$\sum_{n=2}^{\infty} \frac{24}{(n-1)^2} = \sum_{n=1}^{\infty} \frac{24}{n^2}$$

(by writing out both sides; or, algebraically, by setting $k = n - 1$ and doing a substitution). Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so does $\sum_{n=1}^{\infty} \frac{24}{n^2}$, so by the comparison test, our original series converges.

4.4.2 Integral test

The argument we used to prove the divergence of the harmonic series also works in general as a test for convergence or divergence of series.

Theorem 4.36. *Suppose* $f\colon [1,\infty) \to \mathbb{R}$ *is a continuous, decreasing, positive function. Then*

$$\sum_{n=1}^{\infty} f(n)$$

converges if and only if

$$\lim_{n\to\infty} \int_1^n f(x)dx$$

converges.

Proof. Since f is decreasing, we can show directly by comparing the area of a rectangles with the area under the curve that

$$f(k+1) \le \int_k^{k+1} f(x)\,dx \le f(k).$$

Therefore for each $n \in \mathbb{N}$ we have

$$\sum_{k=2}^{n+1} f(k) \le \int_1^{n+1} f(x)\,dx \le \sum_{k=1}^{n} f(k).$$

These three sequences are positive and increasing with n.

So if $\sum_{n=1}^{\infty} f(n)$ converges, then its sum is an upper bound to the sequence $\int_1^{n+1} f(x)\,dx$, whence these sequence must converge.

On the other hand, if the sequence $\int_1^{n+1} f(x)\,dx$ converges, then it is an upper bound to the sequence of partial sums of $\sum_{n=2}^{\infty} f(n)$, whence that series converges.

By an exercise, $\sum_{n=2}^{\infty} f(n)$ converges if and only if $\sum_{n=1}^{\infty} f(n)$ converges, whence the result. $\square$

Example 4.37. Using this test, we can show that $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges for each real number $s > 1$, since

$$\int_1^n x^{-s}dx = \frac{1}{-s+1}x^{-s+1}\,|_1^n = \frac{1}{-s+1}n^{-s+1} - \frac{1}{1-s}$$

and since $1 - s < 0$, this converges as $n \to \infty$.

4.4.3 Ratio test

The comparison test works well if you already know about the convergence or divergence of a similar-looking series, and is particularly useful for series with terms which are rational functions.

For series with exponential terms, the ratio and root tests are often more useful.

Theorem 4.38 (Ratio Test)**.** *Suppose* $\{a_n\}_{n\in\mathbb{N}}$ *is a sequence of positive real numbers*[4]*. Suppose*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = q.$$

Then

[4] or, such that there exists $N \in \mathbb{N}$ such that for all $n \ge N$, we have $a_n > 0$

1. *if* $q > 1$ *then* $\sum_{n\in\mathbb{N}} a_n$ *diverges;*

2. *if* $q < 1$ *then* $\sum_{n\in\mathbb{N}} a_n$ *converges;*

3. *if* $q = 1$ *anything can happen.*

Proof. (1) If $q > 1$ then there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\frac{a_{n+1}}{a_n} > 1$$

whence $a_{n+1} > a_n$. But then $\{a_n\}$ is an increasing sequence of positive numbers, which therefore cannot converge to 0. Whence the series diverges.

(2) If $q < 1$ then choose r with $q < r < 1$; so there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\frac{a_{n+1}}{a_n} < r$$

whence $a_{n+1} < ra_n$. This implies that for all $n > N$ we have

$$a_n < r^{n-N} a_N.$$

Now a_N and r^N are constants; we have, for $n > N$, that

$$a_n < \frac{a_N}{r^N} r^n.$$

But since $0 < r < 1$, the series $\sum_{n\in\mathbb{N}} r^n$ converges, whence $\sum_{n\in\mathbb{N}} \frac{a_N}{r^N} r^n$ converges; and so by the comparison test $\sum_{n\in\mathbb{N}} a_n$ converges.

(3) exercise. □

Example 4.39. Let $x \in \mathbb{R}$. For which values of x does the series $\sum_{n=0}^{\infty} \frac{|x|^n}{n!}$ converge?

Solution: This series has positive terms, so we compute

$$\frac{a_{n+1}}{a_n} = \frac{|x|^{n+1}/(n+1)!}{|x|^n/n!} = \frac{|x|}{n+1}$$

which converges to $q = 0$ for all $x \in \mathbb{R}$. So this series converges for every $x \in \mathbb{R}$.

Example 4.40. For which x does the series $\sum_{n=0}^{\infty} \frac{|x|^n}{n+1}$ converge?

Solution: this series has positive terms so we compute

$$\frac{a_{n+1}}{a_n} = \frac{|x|^{n+1}/(n+2)}{|x|^n/(n+1)} = |x|\frac{n+1}{n+2}$$

which converges to $q = |x|$. Thus by the ratio test the series converges if $|x| < 1$ and diverges if $|x| > 1$.

It isn't obvious what happens when $|x| = 1$; but in fact the series diverges at $x = 1$ and converges at $x = -1$.

4.4.4 Root test (optional)

There is a similar test, also proposed by Cauchy.

Theorem 4.41 (Root Test)**.** *Suppose* $\{a_n\}_{n\in\mathbb{N}}$ *is a sequence of positive real numbers*[5]*. Suppose*

$$\lim_{n\to\infty} \sqrt[n]{a_n} = q$$

Then

1. *if* $q > 1$ *then* $\sum_{n\in\mathbb{N}} a_n$ *diverges;*
2. *if* $q < 1$ *then* $\sum_{n\in\mathbb{N}} a_n$ *converges;*
3. *if* $q = 1$ *anything can happen.*

The proof follows a similar model as for the ratio test; it's left as an exercise.

4.4.5 On series with arbitrary terms (optional)

If the terms of the series are not all positive, then you would expect that it is a bit more likely that the series can converge. This is true in the following sense.

Definition 4.42. A series $\sum_{n\in\mathbb{N}} a_n$ is called *absolutely convergent* if the series

$$\sum_{n\in\mathbb{N}} |a_n|$$

is convergent.

What makes this concept attractive is the following.

Theorem 4.43. *If a series is absolutely convergent then it is convergent.*

Idea of proof. Consider the following series:

1. $\sum_{n\in\mathbb{N}} a_n$; denote its partial sums s_n
2. $\sum_{n\in\mathbb{N}} |a_n|$: denote its partial sums s_n^a
3. $\sum_{n\in\mathbb{N}} b_n$, where $b_n = 0$ if $a_n < 0$ and $b_n = a_n$ if $a_n \geq 0$; denote its partial sums s_n^+
4. $\sum_{n\in\mathbb{N}} c_n$, where $c_n = 0$ if $a_n > 0$ and $c_n = a_n$ if $a_n \leq 0$; denote its partial sums s_n^-

[5] or, such that there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $a_n > 0$

Then by hypothesis, $\sum_{n\in\mathbb{N}} |a_n|$ converges; so by the comparison test, $\sum_{n\in\mathbb{N}} b_n$ converges. Since $s_n^- = s_n^+ - s_n^a$ and each of $\{s_n^+\}$ and $\{s_n^a\}$ converge, we deduce that $\sum_{n\in\mathbb{N}} c_n$ converges. Now since $s_n = s_n^+ + s_n^-$, we finally see that $\sum_{n\in\mathbb{N}} a_n$ converges. □

There's lots more to say about the convergence of series, and of how to compute their limits, but we can leave that until MAT2125.

4.4.6 Exercises

1. Prove Proposition 4.28.

2. Prove that if $\sum_{n\geq 0} a_n$ converges, so does $\sum_{n\geq 0} ca_n$ for any constant $c \geq 0$.

3. Prove that if $\sum_{n\geq 0} a_n$ diverges, so does $\sum_{n\geq 0} ca_n$ for any constant $c > 0$.

4. Let $N \in \mathbb{N}_+$. Prove that $\sum_{n\geq 0} a_n$ converges if and only if $\sum_{n\geq N} a_n$ converges. Prove that they are not equal unless $\sum_{n=1}^{N-1} a_n = 0$.

5. Prove that if $\sum_{n\geq 0} a_n$ and $\sum_{n\geq 0} b_n$ converge, then so does $\sum_{n\geq 0}(a_n + b_n)$, and that $\sum_{n\geq 0}(a_n + b_n) = \sum_{n\geq 0} a_n + \sum_{n\geq 0} b_n$.

6. Prove that if you group the terms of the series $\sum_{n=1}^{\infty}(-1)^n$ in pairs, you can get a sum of 1, or else a sum of 0. Explain why this is equivalent to showing that the sequence of partial sums of this divergent series has some subsequences which converge to 0 and to 1.

7. Compare $\left(\sum_{n=0}^{\infty} \frac{1}{2^n}\right)\left(\sum_{n=0}^{\infty} \frac{1}{3^n}\right)$ and $\sum_{n=0}^{\infty} \left(\frac{1}{2^n}\right)\left(\frac{1}{3^n}\right)$. Conclude that the product of two series is not given by taking the series of the product of the terms.

8. Consider the series with positive terms $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+4}{5n^3+42n+1}$. What is the degree of its terms (that is, what is the degree of the numerator (as a polynomial in n) minus the degree of the denominator (as a polynomial in n))? Given this, do you expect it to converge or to diverge? For a comparison test, which direction of inequality should you try to get?

9. Same, for $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+4}{5n^4+42n+1}$.

10. (challenging exercise) The *Riemann zeta function* is a function $\zeta\colon \mathbb{C} \to \mathbb{C}$ which, for real numbers $s > 1$ is defined by
$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}.$$
(This is the function whence the infamous Riemann hypothesis springs.) Prove, using an argument similar to what we used for the harmonic series ($s = 1$), that $\zeta(s)$ converges for $s > 1$. This time, however, you'll want to bound the terms above so that you get an upper bound on the series, hence convergence. (By the comparison test, you can also prove it diverges for $s < 1$.)

11. Use the integral test to prove that $\sum_{n=1}^{\infty} n^{-4/3}$ converges and $\sum_{n=1}^{\infty} n^{-3/4}$ diverges.

12. Prove that the sequence of quotients $\{a_{n+1}/a_n\}_{n\geq 1}$ of the terms of the each of the following series (with positive terms) converges to 1. Conclude case (3) of the Ratio test:

(a) The (divergent!) harmonic series $\sum_{n\geq 1}\frac{1}{n}$

(b) The convergent series $\sum_{n\geq 1}\frac{1}{n^2}$

13. Apply the ratio test to each of the following series to decide if they converge or not.

(a) $\sum_{n\geq 0}\frac{n+1}{n!}$

(b) $\sum_{n\geq 1}\frac{n^3}{e^n}$

14. Apply the ratio test to each of the following series to decide for which $x\in\mathbb{R}$ they converge:

(a) $\sum_{n\geq 0}\frac{1}{n!}|x|^n$

(b) $\sum_{n\geq 1}\frac{1}{n}|x|^n$

15. Prove the root test. Note that $0<\sqrt[n]{a_n}<r<1$ implies $0<a_n<r^n$ and $1<r<\sqrt[n]{a_n}$ implies $1<r^n<a_n$; so a comparison test with the geometric series is possible in each case.

4.5 Back to Taylor polynomials, and Taylor series

4.5.1 Taylor series

So our work with Taylor polynomials begs the question: why stop with polynomials? Why not just take the whole series?

Theorem 4.44. *Let $R_n(x)$ be the Taylor remainder of f at a and let $r>0$ (or "$r=\infty$"). If $\lim_{n\to\infty}R_n(x)=0$ for all x such that $|x-a|<r$ then*

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

for all $x\in(a-r,a+r)$. This series is called the Taylor series of f centered at a*. If r is maximal such that this equality holds we call it the* radius of convergence of the Taylor series of f centered at a*. If it holds for all $r>0$ then we say the radius of convergence is ∞.*

So for example, by our calculations above we can conclude that

$$e^x=1+x+\frac{1}{2}x^2+\frac{1}{3!}x^3+\cdots=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

for all $x\in\mathbb{R}$ (so the radius of convergence is ∞); and also that

$$\ln(x)=(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\cdots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^n}{n}$$

for all $x\in(\frac{1}{2},\frac{3}{2})$ (so the radius of convergence is at least $\frac{1}{2}$).

Remark 4.45. It is possible for a function to have radius of convergence 0, meaning roughly that the approximation is horrible; see the exercises.

Example 4.46. Show that $\sin(x)$ is equal to its Taylor series (centered at 0) for all x.

Solution: we compute

	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}x^n$
$f(x)$	$\sin(x)$	0	0
$f'(x)$	$\cos(x)$	1	x
$f''(x)$	$-\sin(x)$	0	0
$f'''(x)$	$-\cos(x)$	-1	$-\frac{1}{3!}x^3$
$f^{(4)}(x)$	$\sin(x)$	0	0

and we see the pattern; the Taylor series for $\sin(x)$ is

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!}x^{2n+1}.$$

Since the $n+1$st derivative is bounded by $M = 1$, we see as we did for e^x that the Taylor remainder goes to 0 as $n \to \infty$, for any r. So $\sin(x)$ is equal to its Taylor series.

The real interest with Taylor series is that sometimes we can find them without having to compute all the derivatives at a point — what's really going on is that some functions (including most functions of interest to us in Calculus) are *analytic*, meaning, equal to a "power series", and in fact, many more functions for which we don't have formulae are expressible as their Taylor series.

Definition 4.47. A sum of the form $\sum_{n=0}^{\infty} b_n(x-a)^n$, where x is a real variable, is called a *power series*. If it converges for all x then we say it has radius of convergence ∞. Otherwise, there exists $r \geq 0$ such that it converges for all x with $|x-a| < r$ and diverges for all x with $|x-a| > r$; in this case we say its radius of convergence is r.

Power series have excellent properties, as the following proposition shows.

Proposition 4.48. *Let $r > 0$. Suppose $\sum_{n=0}^{\infty} b_n(x-a)^n$ is a power series with radius of convergence r. Define the function $f\colon (a-r, a+r) \to \mathbb{R}$ by*

$$f(x) = \sum_{n=0}^{\infty} b_n(x-a)^n.$$

Then:

(a) the power series coincides with the Taylor series of f centered on a, on $(a-r, a+r)$;

(b) $f'(x) = \sum_{n=0}^{\infty} nb_n(x-a)^{n-1}$; and

(c) $\int_a^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{1}{n+1} b_n(x-a)^{n+1}$;

and the radii of convergence of each of the series in (b) and (c) are also r.

Therefore, if you have a function which you know by any means is expressible as a power series, then in fact that is its Taylor series.

Example 4.49. Find the Taylor series of $\cos(x)$.

We have $\cos(x) = \frac{d}{dx}\sin(x)$ and $\sin(x) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!}x^{2n+1}$ for all $x \in \mathbb{R}$ so by Proposition 4.48

$$\begin{aligned}
\cos(x) &= \frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!}x^{2n+1}\right) \\
&= \sum_{n=0}^{\infty}\frac{d}{dx}\left((-1)^n \frac{1}{(2n+1)!}x^{2n+1}\right) \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n)!}x^{2n} \\
&= 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \cdots
\end{aligned}$$

which Proposition 4.48 promises must coincide with the Taylor series of $\cos(x)$ centered at 0.

Compare this with our calculation in Example 4.17.

Sometimes, you know the Taylor series of a function already.

Example 4.50. Recall the geometric series

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}$$

which converges for any $|r| < 1$. Replace r with x and you get

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

and so by Proposition 4.48 this is in fact the Taylor series for $\frac{1}{1-x}$ centered at 0.[6] The radius of convergence of this series is 1.

Example 4.51. (Using a substitution) Find the Taylor series for $\frac{1}{1+x^2}$.

Rather than working out derivatives, notice that

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

so we have

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

which converges for $|-x^2| < 1$, which implies, for $|x| < 1$.

[6]If this seems like a leap of faith: take the nth derivative of both sides at $a = 0$ and agree that you're proving the Taylor polynomial coefficients are all 1.

Example 4.52. Find the Taylor series for arctan(x).

We know

$$\arctan(x) = \int_0^x \frac{1}{1+t^2} dt$$

so since $\frac{1}{1+t^2} = \sum_{n=0}^{\infty}(-1)^n t^{2n}$, integrating term by term yields

$$\arctan(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

which has radius of convergence 1.

Remark 4.53. We do not *a priori* know if the series converges or diverges at $x = r$. In the case of arctan(x), the series does converge for $x = 1$, yielding a wonderful formula for π:

$$\pi/4 = \arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

although there exist many others similar Taylor approximations one can use.

So functions that have no simple anti-derivative may still have a Taylor series (which means they can be integrated term by term).

Example 4.54. Consider e^{-x^2}. Its Taylor series is obtained from that of e^x as

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

so you can integrate this term by term to solve for

$$\int_0^x e^{-t^2} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} x^{2n+1},$$

which in many ways is a lot nicer than numerical integration, because at least it's a formula.

Remark 4.55. (added after class) Under differentiation, integration, and multiplication by a polynomial, the radius of convergence of your series does not change, and neither does its centre. If you make a substitution, both can change. For example, the series for

$$\frac{1}{1+(2x+5)} = \sum_{n=0}^{\infty}(2x+5)^n = \sum_{n=0}^{\infty} 2^n (x + \frac{5}{2})^n$$

was obtained by making a substitution $u = 2x+5$ in a series centered at 0 with radius of convergence 1; it is now centered at $-\frac{5}{2}$ and has radius of convergence $\frac{1}{2}$. This is because the series converges for $|u| < 1$, which means $|2x + 5| < 1$, which is equivalent to $-3 < x < -2$.

4.5.2 Exercises

1. Find the Taylor series of $\cos(x)$ centered at 0 directly, by computing its derivatives.

2. Let f be defined as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ e^{-x^{-2}} & \text{if } x > 0 \end{cases}$$

   (a) Verify that f is differentiable at 0 and that $f'(0) = 0$. Hint: show that this comes down to proving that

$$\lim_{x \to 0} \frac{e^{-x^{-2}}}{x} = 0$$

   which, since this is an indeterminate form of type 0/0, you can prove using l'Hospitals rule.

   (b) Similarly $f^{(n)}(0) = 0$ for all $n \ge 1$. Hint: after part (a) you know that

$$f'(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 2x^{-3}e^{-x^{-2}} & \text{if } x > 0 \end{cases}$$

   so you work out $f''(0)$ as above. Nonetheless, there is a lot of hard work involved!

   (c) Write down the Taylor series for f and explain why its radius of convergence is 0, that is, f is not equal to its Taylor series on any open interval centered at 0.

   This is an example of a non-analytic function, that is, one which is not equal to its Taylor series anywhere except at a.

3. Use the methods in the latter part of this section to find the Taylor series of $\frac{1}{1-4x}$. What is the radius of convergence of this series? Now use these ideas to give an example of a Taylor series with radius of convergence equal to 4.

4. [S]: Section 8.6 #3,5,7,23,25

5. [S]: Section 8.7 #5, 7, 11, 13, 19, 29, 33, 53, 59, 61

6. It turns out that Taylor series converge exactly on an interval of the form $(a-r, a+r)$ (or else on all of $\mathbb{R}$, or else only at a). With this in mind, solve [S] Section 8.5 # 5, 13, 19.

7. Find a Taylor series expansion for each of the following functions, given the Taylor series in Table 1, Chapter 8.7 of Stewart (also those derived in class). In each case, give the point a at which the Taylor series is centered, and give the radius of convergence.

   (a) $f(x) = \cos(-x^2)$

   (b) $g(x) = \dfrac{x^2}{2+x}$ (Hint: $2 + x = 2(1 + x/2)$)

   (c) (bonus) Given that i is a complex number which satisfies $i^2 = -1$, write down the Taylor series for e^{ix} and verify it equals that of $\cos(x) + i\sin(x)$.

Chapter 5

Integrals

5.1 Applications of integration

5.1.1 Recall the definition of the integral

The material in this section comes from [S, Section 5.1].

In MAT1320, we encountered the notion of the (definite) integral of a function. Namely, if f is a function defined on $[a, b]$, then

$$\int_a^b f(x)\, dx$$

measures the (signed) area of the region cut out by $x = a$, $x = b$, $y = f(x)$ and $y = 0$; and it is defined as follows.

For any n, divide the interval $[a, b]$ into n subintervals of width

$$\Delta x = \frac{b-a}{n}.$$

Denote the endpoints of these subintervals by $x_0 < x_1 < \cdots < x_n$, and for each index $i \in \{1, 2, \cdots, n\}$, choose a sample point $x_i^* \in [x_{i-1}, x_i]$. Then consider the rectangle of height $f(x_i^*)$ over this subinterval; it has (signed) area $f(x_i^*)\Delta x$. The sum of the areas of these rectangles is the *Riemann sum*

$$S_n = \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

In this way, one can form a sequence of Riemann sums (eg, the sequence of left Riemann sums, the sequence of right Riemann sums, the sequence of midpoint rule Riemann sums). If all of these sequences converge to the same value then we say f is integrable and we define

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} S_n.$$

Remark 5.1. Note that although each Riemann sum is a sum, this is not a *series* that we are dealing with: to get from S_n to S_{n+1} you don't just add another term — you have to recalculate everything! It's just a sequence in which each of the terms is quite complicated and given by a sum.

This definition seems quite unusable, since it asks us to verify that all sequences of Riemann sums converge to the same value, and given how hard it is to come up with a formula for even one sequence of Riemann sums, this is quite daunting. A big part of MAT2125 is thinking through how to make this reasonable, and to gain a deeper understanding of integrability.

So instead, let's take this in a different direction: given the concept of integration as the limit of sums of areas of rectangles to produce the area of an irregular region, what else can we compute with it?

5.1.2 Area between curves

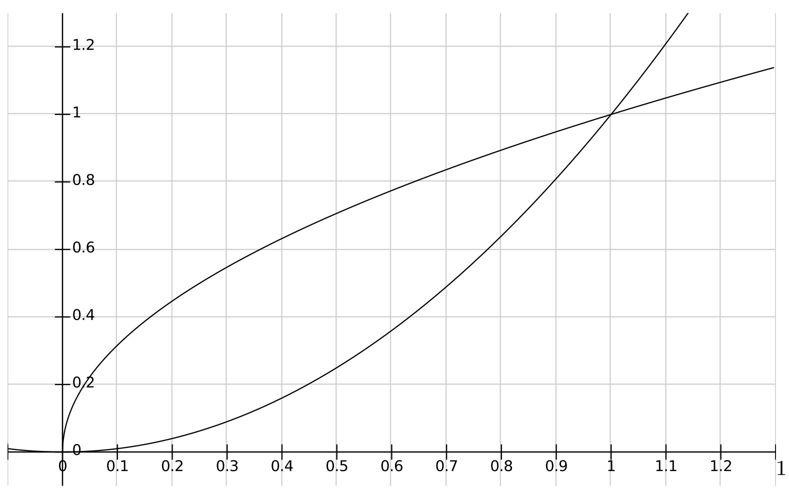
The material in this section comes from [S, 6.1].

Now let us generalize to find the area of a region which is not flat on 3 sides.

Example 5.2. Say we wanted to find the area enclosed by the two curves

$$y = \sqrt{x} \quad \text{and} \quad y = x^2.$$

We sketch these curves and see that indeed they cut out a region in the first quadrant.



What is the area of this region? Let's use the same ideas as before. Note thatthe extrema of the region are $x = 0$ and $x = 1$ (as we compute algebraically as the intersection points of the two curves). Divide this interval $[0, 1]$ into n subintervals and draw a rectangle over each subinterval, in such a way that the bottom edge of the rectangle touches the curve $y = x^2$ and the top edge touches the curve $y = \sqrt{x}$. Summing the areas of all these rectangles should give an approximation of the area of the region.

[1]Image created by the online graphing utility FooPlot.

So what is the height of such a rectangle? Well, if we take a sample point x_i^* in $[x_{i-1}, x_i]$ then we could select $\sqrt{x_i^*}$ for the top edge and $(x_i^*)^2$ for the bottom edge, yielding

$$S_n = \sum_{i=1}^{n} (\sqrt{x_i^*} - (x_i^*)^2)\Delta x.$$

But this is the same thing as a Riemann sum for the integral of $\sqrt{x} - x^2$! Therefore, since $\sqrt{x} - x^2$ is integrable, if we take the limit as $n \to \infty$ of these sums S_n our answer will be

$$\int_0^1 (\sqrt{x} - x^2)\, dx.$$

So we conclude that the area of this region is

$$\begin{aligned} \int_0^1 (\sqrt{x} - x^2)\, dx &= \left[\frac{1}{3/2}x^{3/2} - \frac{1}{3}x^3\right]_0^1 \\ &= \frac{2}{3} - \frac{1}{3} - (0) = \frac{1}{3}. \end{aligned}$$

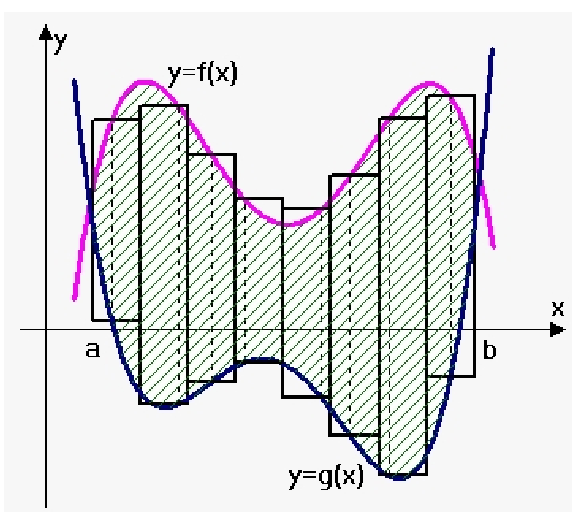
That's quite satisfying; we know that $\int_0^1 x^2\, dx = \frac{1}{3}$ and so conclude that these curves cut the unit square into 3 equal-sized regions.

Lesson 1: So generally speaking, given a region which is bounded by

$$\begin{aligned} x &= a && \text{on the left} \\ x &= b && \text{on the right} \\ y &= f(x) && \text{on the top} \\ y &= g(x) && \text{on the bottom} \end{aligned}$$

then the area of this region is

$$\int_a^b (f(x) - g(x))\, dx.$$

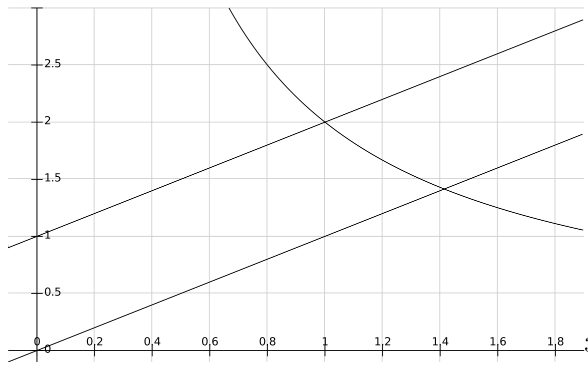


[2]

[2]Image taken from the website emathhelp.net.

Example 5.3. What is the area of the region bounded by $y = 2/x$, $y = x$ and $y = x + 1$ in the first quadrant?

We sketch this region and it's a tilted four-sided region. We calculate and mark the four "corners" or intersection points: $(0,0)$, $(0,1)$, $(\sqrt{2}, \sqrt{2})$ and $(1,2)$.



[3]

When $0 \le x \le 1$, the top curve is $y = x + 1$ and the bottom curve is $y = x$. When $1 \le x \le \sqrt{2}$, the top curve is $y = 2/x$ and the bottom curve is $y = x$. Therefore we have to split our integral at $x = 1$. The area will be

$$\begin{aligned} A &= \int_0^1 ((x+1) - x)dx + \int_1^{\sqrt{2}} (\frac{2}{x} - x)dx \\ &= \int_0^1 1dx + \left[2\ln(x) - \frac{1}{2}x^2\right]_1^{\sqrt{2}} \\ &= 1 + (2\ln(\sqrt{2}) - \frac{1}{2}(2)) - (2\ln(1) - \frac{1}{2}) \\ &= 1 + \ln(2) - 1 - 0 + \frac{1}{2} = \frac{1}{2} + \ln(2). \end{aligned}$$

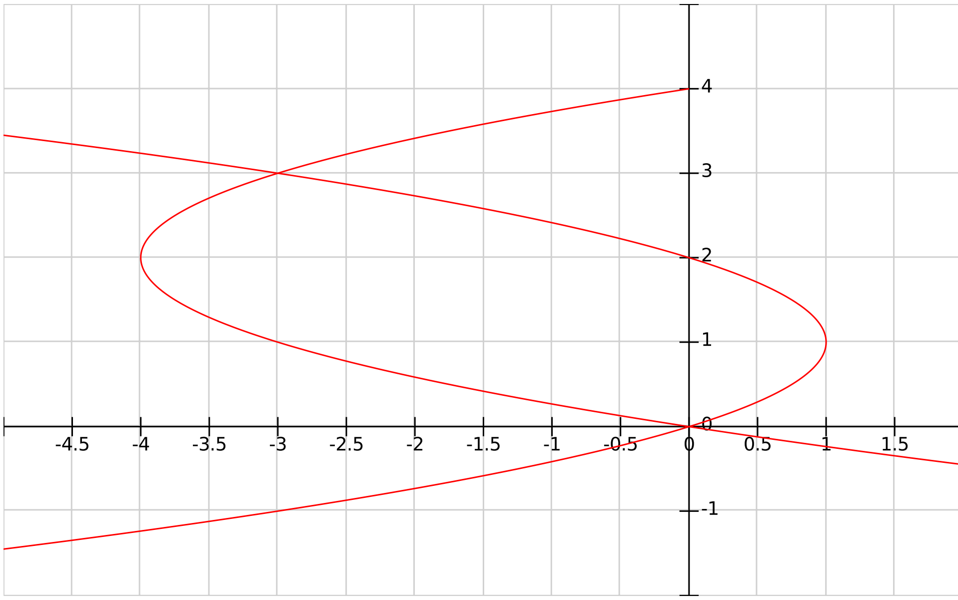
Lesson 2: The "top" or "bottom" function may be given by a different formula over different intervals. Split your integral accordingly, so that on each interval, you have one clear top curve and one clear bottom curve.

But we can be even more general than this.

Example 5.4. Find the area of the region enclosed by $x = y^2 - 4y$ and $x = 2y - y^2$.

We sketch the region. Note that $x = y(y-4)$ is a parabola opening to the right with intercepts at $(0,0)$ and $(0,4)$, whereas $x = y(2-y)$ is a parabola opening to the left with intercepts at $(0,0)$ and $(0,2)$. So they bound a funny region. The points of intersection are $(0,0)$ and $(-3,3)$.

[3]Image created by the online graphing utility FooPlot.



[4]

Oh, what a pain this one is! There are three different intervals on the x-axis to consider, where the top curve and bottom curve are different, so this would be the sum of three integrals (exercise).

But while we procrastinate on doing this, we notice that if we turned the page by $\pi/2$, the problem is a lot easier to solve. So it *should* be easy to solve, and indeed it is.

Let's go back to our original concept, and this time, let's subdivide the interval $[0, 3]$ on the y-axis into n subintervals, each of width Δy. On each subinterval, draw a horizontal rectangle with left side on the curve $g(y) = y^2 - 4y$ and right side on the curve $f(y) = 2y - y^2$. Taking a sample point y_i^* in the subinterval to choose precisely where to put the ends of the rectangles, we get that the sum of the areas of these rectangles is

$$S_n = \sum_{i=1}^{n} (f(y_i^*) - g(y_i^*))\Delta y.$$

Taking the limit as $n \to \infty$ yields $\int_0^3 (f(y) - g(y))dy$. That is,

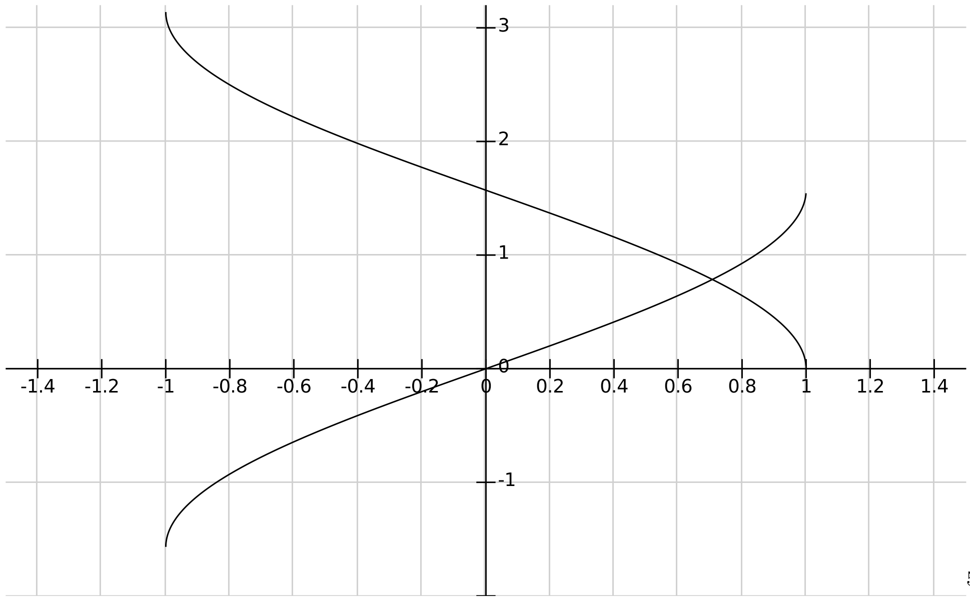
$$\begin{aligned}\int_0^3 (2y - y^2) - (y^2 - 4y)dy &= \int_0^3 (-2y^2 + 6y)dy \\ &= \left[-\frac{2}{3}y^3 + 3y^2\right]_0^3 \\ &= -\frac{2}{3}(27) + 27 - 0 = \frac{1}{3}(27) = 9.\end{aligned}$$

Lesson 3: You can integrate with respect to y if it's more convenient. Do not change the names of any variables! Instead, get used to the idea that your variable of integration need not always be x — something that will be important to get used to before doing multivariable integration next year.

[4] Image created by the online graphing utility FooPlot.

Example 5.5. What is the area of the region bounded by the curves $y = \arcsin(x)$, $y = \arccos(x)$ and $x = 0$?

Well, we have to start by sketching this region.



[5]

Recall[6] that $y = \arcsin(x)$, being the inverse sine function, has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$; it is increasing on that interval and has a vertical tangent at each endpoint. On the other hand, $y = \arccos(x)$ has domain $[-1, 1]$ and range $[0, \pi]$ and is *decreasing*.

We therefore see that there is a roundish triangular region in the first quadrant enclosed by the 3 curves ($x = 0$ being the third).

To find the area, we need the upper limit of integration, which is the point of intersection of the two curves. This intersection is the point x for which $\arcsin(x) = \arccos(x)$ and is $\sqrt{2}/2$.[7]

Good; we now know that the area of the region is

$$\int_0^{\sqrt{2}/2} (\arccos(x) - \arcsin(x))\, dx$$

Oh. That's unfortunate; this is quite a bit of work to integrate, using integration by parts (exercise).

Well, so let's instead turn it around. We get

$$\int_0^{\pi/4} \sin(y)\, dy + \int_{\pi/4}^{\pi/2} \cos(y)\, dy$$

[5]Image created by the online graphing utility FooPlot.

[6]See [S, Section 3.6] for more details.

[7]If this doesn't seem obvious, write the curves as $\sin(y) = x$ and $\cos(y) = x$; then the point of intersection is where $\sin(y) = \cos(y)$ which you know is $y = \pi/4$, and at $y = \pi/4$ we get $x = \sqrt{2}/2$.

which gives

$$[-\cos(y)]_0^{\pi/4} + [\sin(y)]_{\pi/4}^{\pi/2} = (-\sqrt{2}/2 - (-1)) + (1 - \sqrt{2}/2) = 2 - \sqrt{2}$$

and that was a lot easier.

5.1.3 Exercises

1. Compute the area of the region in Example 5.4 by dividing it into 3 subintervals and integrating over x. You have to find the top and bottom functions, and the points of intersection, to do this.

2. Compute the area of the region in Example 5.5 as a integral over x, as originally derived, using integration by parts.

There are many excellent exercises on this topic, and on the subject of the following sections, in the textbook by Stewart [S].

5.2 More applications of integration

5.2.1 Average value of a function

The material in this section comes from [S, 6.5].

Given a finite set of numbers $\{a_1, \cdots, a_n\}$, we define their average as

$$a = \frac{1}{n}\sum_{i=1}^{n} a_i.$$

This formula makes no sense if the set is infinite, even though there are circumstances where the notion of "average value" still makes sense.

For example, the water level on a lake varies continuously over a year, but we still have a notion of its "average level". Let's think about how we compute this.

Suppose we have a function f defined on an interval $[a, b]$. We want to know its average value over this interval. We could subdivide $[a, b]$ into n subintervals of length Δx, and by choosing a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$, we approximate the average value of f by the average value of f at all these sample points, that is,

$$Av_n = \frac{1}{n}\sum_{i=1}^{n} f(x_i^*).$$

This doesn't quite look like a Riemann sum yet. But recall that

$$\Delta x = \frac{b-a}{n}$$

so in fact we have

$$Av_n = \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Excellent! Taking the limit as $n \to \infty$ (which corresponds to sampling more and more points and thus, intuitively, giving the average value of the function) yields the formula

$$Av(f) = \int_a^b f(x)\, dx.$$

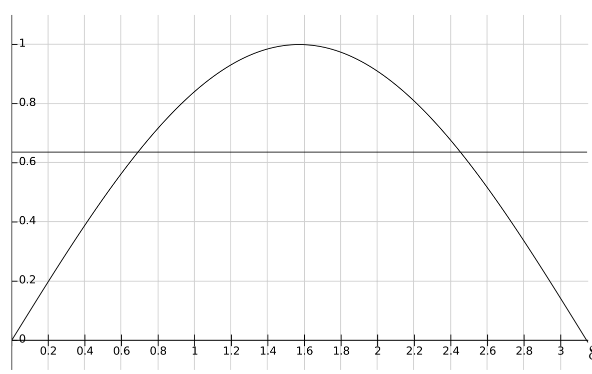
This formula feels quite intuitive: take the area of the region under the curve from a to b and divide by the width of this interval; what you get is a number $Av(f)$ such that the rectangle of height $Av(f)$ and width $b-a$ has the same area as the region under the curve f.

Example 5.6. Find the average value of $f(x) = \sin(x)$ on the interval $[0, \pi]$.

Solution: $\int_0^\pi \sin(x)\, dx = -\cos(x)\Big]_0^\pi = -(-1-1) = 2$ so the average value is

$$\frac{1}{\pi - 0}\int_0^\pi \sin(x)\, dx = \frac{2}{\pi} \sim 0.64$$

which seems reasonable; the area of the rectangle below could certainly be the area under the curve.



[8]

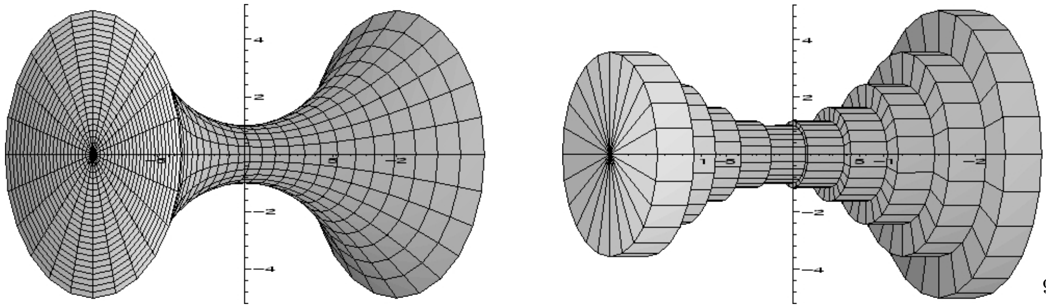
5.2.2 Volumes of 3-dimensional objects

The material in this section comes from [S, Section 6.2 and 6.3].

In MAT2122, you will learn how multivariable integration is used to compute volumes, just as single variable integration computes area. However, for simple regions (that is, ones where we can figure out the cross-sectional area without Calculus), single variable integration suffices.

The idea is the following. Choose an axis and imagine slicing your object like bread, perpendicular to this axis. So the volume of the object is the sum of the volumes of the slices.

[8] Image created by the online graphing utility FooPlot.



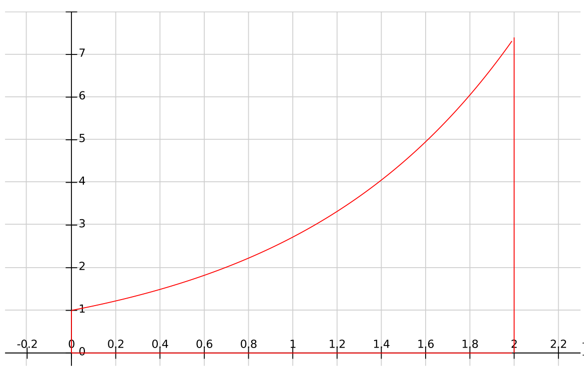
9

If the slices are thin enough, you could approximate their volume as Δx (their width) times the area of some cross-section, call it $A(x_i^*)$. Thus in the limit, the volume should be

$$\int_a^b A(x)\,dx$$

where $A(x)$ is a formula for the cross-sectional area of the object at x.

As an example, consider the region cut out in the first quadrant by $y = e^x$, for $0 \leq x \leq 2$.



10

We could rotate this region around either the x-axis, or the y-axis, to obtain two different solids of revolution; their cross-sectional areas, perpendicular to the axis of rotation, will be either circles or annuli.

Example 5.7. What is the volume of the object obtained by rotating the region *under* the graph of $y = e^x$ between $x = 0$ and $x = 2$ around the x-axis?

We draw a picture. So the axis of rotation is the x-axis, and we'll integrate over x between 0 and 2. At each x, the cross-section is a circle of radius e^x, so the cross-sectional area is

$$A(x) = \pi (e^x)^2 \; units^2.$$

This yields

$$V = \int_0^2 A(x)\,dx = \int_0^2 \pi e^{2x}\,dx = \pi \frac{1}{2} e^{2x}\Big]_0^2 = \frac{\pi}{2}(e^4 - 1) \sim 84.2 \; units^3.$$

[9]Image created by the software program Maple for MA1022, 2009, at the Worcester Polytechnic Institute; it is part of their course materials on line.

[10]Image created by the online graphing utility FooPlot.

So if each cross-section is a disk, then $S = \pi r^2$ where r is the radius of the disk. If your axis of revolution is x then r will be a function of x, because it's the radius at each point x. In this case we write $S(x)$ for the surface area.

Example 5.8. What is the volume of the object obtained by rotating the region *under* the graph of $y = e^x$ between $x = 0$ and $x = 2$ around the y-axis?

We draw a picture. This time the cross-sections (which will be with respect to y are not all circles; above $y = 1$ they look more like washers. Recall that the area of a washer is the area of the outer circle minus the area of the inner circle. Since the inner circle is given by $y = e^x$, the inner radius x is equal to $\ln(y)$. Therefore our cross-sectional area function becomes

$$A(y) = \begin{cases} \pi(2)^2 & \text{if } 0 \le y \le 1 \\ \pi(2)^2 - \pi(\ln(y))^2 & \text{if } 1 \le y \le e^2. \end{cases}$$

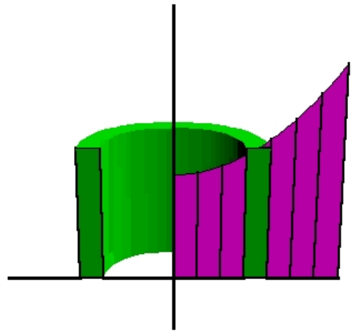
so we have

$$\begin{aligned} V &= \int_0^{e^2} A(y)\, dy \\ &= \int_0^1 4\pi dy + \int_1^{e^2} (4\pi - \pi(\ln(y))^2)\, dy \\ &= \int_0^{e^2} 4\pi dy - \pi \int_1^{e^2} (\ln(y))^2 dy \\ &= 4\pi e^2 - \pi \left(y(\ln(y))^2 \Big|_1^{e^2} - 2 \int_1^{e^2} \ln(y) dy \right) \\ &= 4\pi e^2 - 4e^2\pi + 2\pi \left(y \ln(y) \Big|_1^{e^2} - \int_1^{e^2} dy \right) \\ &= 2\pi(2e^2 - (e^2 - 1)) \\ &= 2\pi(e^2 + 1) \end{aligned}$$

where we used integration by parts twice to solve the integral.

So if each cross-section is an annulus (or call it a washer), then $S = \pi R^2 - \pi r^2$ where R is the outer radius and r is the inner radius. If your axis of revolution is x then in general both R and r will be functions of x and we write $S(x)$ for the surface area.

A different approach which also works: instead of slicing the object and adding up the volumes of the slices, what if we cut the object into concentric cylinders, and add up the volumes of the cylinders? See [S, Section 6.3]. Let's do the same example from this point of view.



[11]

Example 5.9. What is the volume of the object obtained by rotating the region under the graph of $y = e^x$ between $x = 0$ and $x = 2$ around the y-axis?

We draw a picture. Imagine cutting this region into concentric cylinders centered on the y-axis. We approximate the volume of the cylinder by Δx times the surface area $S(x_i^*)$ of the cylinder at distance x_i^*. The surface area of a cylinder of radius x_i^* is given by

$$S(x_i^*) = 2\pi x_i^* h(x_i^*)$$

where $h(x_i^*)$ is the height.

Summing this over all subintervals from the radius a of the innermost cylinder (here $x = 0$) to the radius b of the outermost cylinder (here $x = 2$) gives, in the limit as $n \to \infty$, the answer

$$V = \int_a^b 2\pi x h(x)\, dx$$

In our case the height is the distance between $y = e^x$ and $y = 0$, which are the top and bottom, so $h(x) = e^x - 0 = e^x$ and we have

$$V = \int_0^2 2\pi x e^x\, dx = 2\pi \left[xe^x - e^x \right]_0^2 = 2\pi(2e^2 - e^2 - (-1)) = 2\pi(e^2 + 1)$$

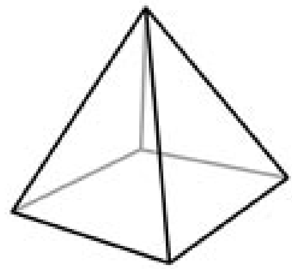
which, thankfully, is the same answer.

So with the method of shells, each little volume is a cylindrical shell, whose volume is approximated by the product of the surface area of the cylinder times the thickness Δx. For a cylinder, the surface area is given by $SA(x) = 2\pi x h(x)$ where $h(x)$ is its height at x. This time, you are integrating on an axis *perpendicular* to the axis of rotation (for example, if you rotate around the x-axis then your integral will be in y, because your shells are moving away from the axis).

One can apply the same principles to compute the volume of any sufficiently regular figure, even if we aren't given explicit functions describing its boundaries.

Example 5.10. Consider a pyramid of height 5, with base a square with side length 2. We put it so it is standing on the xy-plane and centered on the z-axis.

[11] This image comes from the website mathdemos.org.



[12]

Let's cut the pyramid into slices perpendicular to the z-axis. So z runs from $a = 0$ to $b = 5$. At each height z, the cross-section of the pyramid is a square. What are the dimensions of this square? They depend on z; it will be a function $A(z)$.

We draw a picture. In this case, let's draw just the cross-section with the xz plane, which will help us figure out the dimensions of the square. By similar triangles,

$$\frac{x}{5-z} = \frac{1}{5}$$

so $x = (5-z)/5$. Therefore the square has side-length $2x = 2(5-z)/5$, yielding our formula

$$A(z) = 4(5-z)^2/25.$$

We quickly check that this is reasonable; at $z = 0$ we get $A(0) = 4$, which is indeed the area of the base; and at $z = 5$ we get $A(5) = 0$, which is indeed the area of the tip.

So the volume of the pyramid is

$$V = \int_0^5 \frac{4}{25}(5-z)^2 \, dz = \frac{4}{25}\int_5^0 u^2(-1)du$$

(where we did a substitution $u = 5 - z$, $du = -dz$)

$$= \frac{4}{25}(-1)\frac{1}{3}u^3\Big]_5^0 = \frac{4}{75}(0-(-125)) = \frac{1}{3}(20)$$

which coincides with the known formula of $\frac{1}{3}b^2h$.

5.3 A reminder that limits aren't all they seem

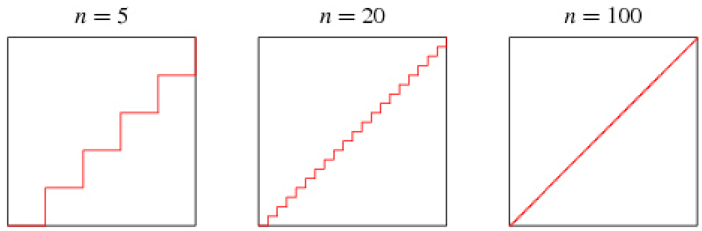
5.3.1 Arc length of a curve

The material in this section comes from [S, Chapter 6.4].

So far we have spoken of areas and volumes, which are 2- and 3- dimensional measures, respectively. What of the length of a curve (a 1-dimensional measure)? Here are some examples to suggest that this is a more delicate problem.

[12] Image obtained from KidsMathGamesOnline.

Example 5.11. The diagonal paradox:



[13]

"Obviously" the curves are getting closer and closer to the diagonal. But since each curve is composed of line segments going only vertically and horizontally, the sum of the lengths of these segments never changes: it's always 2. But the length of the diagonal is $\sqrt{2}$.

Oops. Well, fine, so of course we should approximate the arc length using secants of the curve, that is, line segments that connect nearby points on the curve. Is that enough?

Example 5.12. The paradox of the coast of England:


[14]

We'd like to measure the length of England's coastline. So we start with line segments that are 100km long, and get one estimate. Then we use shorter line segments, to get a better estimate — but instead of getting smaller, the number gets larger! In fact, it's clear that the shorter your line

[13] Weisstein, Eric W. "Diagonal Paradox." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/DiagonalParadox.html

[14] "Britain-fractal-coastline-100km". Licensed under CC BY-SA 3.0 via Wikimedia Commons - http://commons.wikimedia.org/wiki/File:Britain-fractal-coastline-100km.png#/media/File:Britain-fractal-coastline-100km.png; "Britain-fractal-coastline-50km". Licensed under CC BY-SA 3.0 via Wikimedia Commons - http://commons.wikimedia.org/wiki/File:Britain-fractal-coastline-50km.png#/media/File:Britain-fractal-coastline-50km.png

segment, the longer the supposed coastline, because of its infinite twists and beds. In fact, we have to conclude the coastline is infinite!

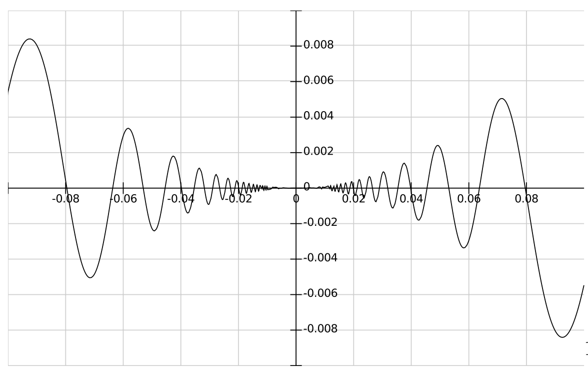
In fact, mathematical examples of the above analogy exist (and are an example of fractals). The *Koch snowflake* is such an example, although to be precise it is just a curve and not the graph of a function. A more complicated example is the *Weierstrass function* which is continuous everywhere but differentiable nowhere; it also has infinite arc length.

Oops. Fine, the coastline of England is very not differentiable, it's more like a fractal than a nice function. And of course since we are approximating with secant lines, we should ask for the function to be differentiable at a minimum. Will that solve the problem?

Example 5.13. The curve

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

is differentiable at all x, but its derivative is not continuous at 0 :



[15]

It is possible to show that the arc length of this curve, between -1 and 1, is infinite.

Upshot: our method for approximating the arc length of a curve will work when the curve is *smoot*: it has a continuous derivative at every point. We call such a function *of class* C^1, and these are the rectifiable functions.

The concept we start with is as follows. Given the graph of $y = f(x)$, we wish to know the length of the curve between $x = a$ and $x = b$. We proceed as usual, by subdividing the interval $[a, b]$ into n subintervals of equal width Δx. Now let us approximate the length L_i of the bit of curve over $[x_{i-1}, x_i]$.

So the line segment L_i is the hypotenuse of a triangle of width Δx and height $\Delta y = f(x_i) - f(x_{i-1})$. By the Mean Value Theorem, there is some point x_i^* in the subinterval for which

$$f'(x_i^*) = \frac{\Delta y}{\Delta x}.$$

[15] Cet image a été crée par le logiciel en-ligne FooPlot.

(And note that since f' is continuous, f' achieves its maximum and its minimum at some (other) sample points in the interval; there can't be an interval on which f' gets arbitrarily far away from the sample value above.)

So the length is

$$\begin{aligned} L_i &= \sqrt{(\Delta x)^2 + (\Delta y)^2} \\ &= \sqrt{(\Delta x)^2 \left(1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right)} \\ &= \sqrt{1 + f'(x_i^*)^2}\Delta x \end{aligned}$$

whence our approximation to the length of the curve is

$$L_n = \sum_{i=1}^{n} \sqrt{1 + f'(x_i^*)^2}\Delta x$$

which we recognize as a Riemann sum; therefore as $n \to \infty$ this converges to

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

This is our formula for the arc length.

Unfortunately, these integrals are often difficult, if not impossible, to evaluate!

Example 5.14. Find the circumference of a circle of radius r.

We'll do portion in the first quadrant, which is given by $f(x) = \sqrt{r^2 - x^2}$ between $x = 0$ and $x = r/\sqrt{2}$; the function is differentiable on this closed interval and it represents 1/8 of the circumference. So

$$f'(x) = \frac{1}{2}(r^2 - x^2)^{-1/2}(-2x) = -x(r^2 - x^2)^{-1/2}$$

which gives

$$1 + f'(x)^2 = 1 + \frac{x^2}{r^2 - x^2} = \frac{r^2}{r^2 - x^2} = \frac{1}{1 - (x/r)^2}$$

so that

$$L = \int_0^{r/\sqrt{2}} \frac{1}{\sqrt{1 - (x/r)^2}} dx$$

which we can solve with the trigonometric substitution $x/r = \sin(\theta)$; thus $\frac{1}{r}dx = \cos(\theta)d\theta$ and if $x = 0$ then $\theta = 0$ but when $x = r/\sqrt{2}$, we have $\theta = \pi/4$. Thus

$$L = \int_0^{\pi/4} \frac{r\cos(\theta)}{\sqrt{1 - \sin^2\theta}} d\theta = r\int_0^{\pi/4} \frac{\cos\theta}{\cos\theta} d\theta = r\pi/4.$$

as required.

5.3.2 Exercises

Review the examples in the textbook and tackle some of the exercises. The idea is not to memorize any particular formula, but rather to understand how we can approach a problem and use our solid understanding of Calculus to find a solution.

Chapter 6

Curves and surfaces in 2 and 3 dimensions

Our next goal, and that which will occupy us for the rest of the course, is understanding differential Calculus in 3 (and more) dimensions. This topic will be continued in MAT2122, where you also learn about multi-variable integral Calculus, and the astonishing, amazing, complicated-but-in-the-right-light-totally-simple generalizations of the Fundamental Theorem of Calculus.

In linear algebra, you have become familiar with the analogy between lines in $\mathbb{R}^2$ and planes in $\mathbb{R}^3$, and know that a plane in $\mathbb{R}^3$ has the form

$$\{(x, y, z) \mid ax + by + cz = d\}$$

for some constants a, b, c, d.

As a way of easing into more complex graphs, let's consider conic sections in $\mathbb{R}^2$ (given by quadratic equations) and their analogues, the quadric surfaces, in $\mathbb{R}^3$.

6.1 Geometry of conic sections in $\mathbb{R}^2$ and their surfaces of revolution

(See [S, Appendix B].)

Among all curves in $\mathbb{R}^2$, the *conic sections* play a particularly important role. These are the curves that can arise by intersecting a double cone with the plane; see Figure 6.1.

6.1.1 The parabola

The parabola can be defined geometrically by starting with a point (called the focus) and a line (not going through the point, called the directrix). The parabola is the set of all points equidistant from the focus and the directrix. (See [S, Appendix B].) A consequence of this defining property is

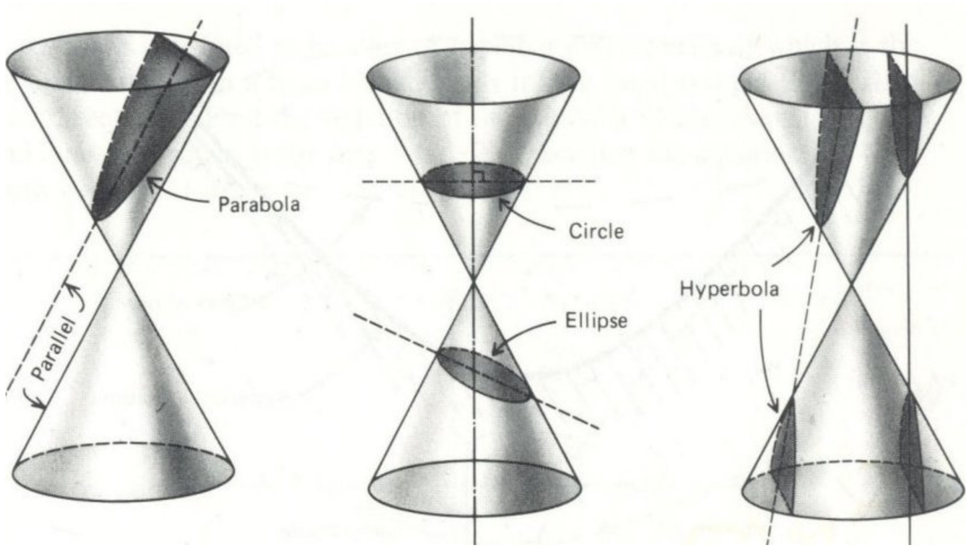
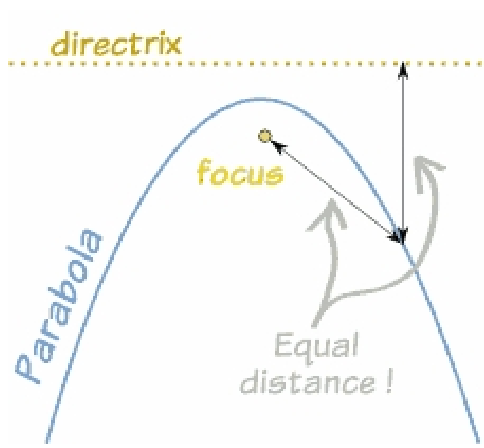
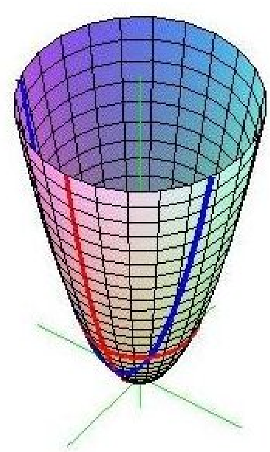


Figure 6.1: This figure of conic sections taken from the following website, where you can also find more about conic sections: `http://www.andrews.edu/~calkins/math/webtexts/numb19.htm`.

that light rays hitting the parabola (on the inside) perpendicular to the directrix will reflect off the curve and pass through the focus. Thus parallel light rays focus onto the focus, making this shape (or rather, that of the corresponding surface of revolution) appropriate for mirrors in telescopes, for example.



[1]


[2]

One can prove (exercise) that by choosing coordinates appropriately, every parabola has the form

$$y = ax^2$$

for some constant a.

[1]This image of the geometric definition of a parabola is from the website mathisfun.org.

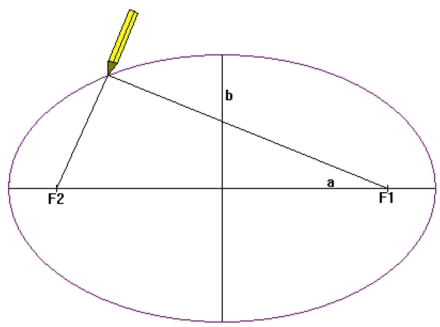
[2]This image of a paraboloid taken from: The Worlds of David Darling, Encyclopedia of Sciences, www.daviddarling.info

If $a = 0$ it is a degenerate parabola; in this case a line.

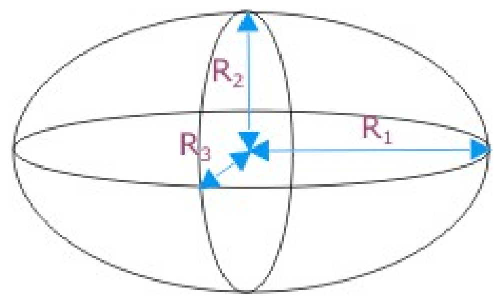
The surface of revolution obtained by rotating the parabola around its central axis (y-axis, here) is called a *paraboloid*.

6.1.2 The ellipse

The ellipse can be defined geometrically be starting with two foci and tracing the set of points with the property that the sum of their distances from the foci is a constant. This is achieved, for example, by attaching a string of fixed length to the two focii and tracing the shape you get by stretching the string to a rigid pair of straight line segments. A consequence of this defining property is that light rays emanating from one focus will reflect off the surface and arrive at the other focus; consequently, this shape (or rather, that of the corresponding surface of revolution) is appropriate for use in medical procedures that require targetting a point on the inside of the body with a source that is highly dissipative, like sound.



[3]



[4]

One can prove (exercise) that by choosing coordinates appropriately, every ellipse has the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

for some constants $a, b > 0$. This ellipse passes through the points $(\pm a, 0)$ and $(0, \pm b)$. If $a \geq b$ then its foci are at the points $(\pm c, 0)$ where $c = \sqrt{a^2 - b^2}$; otherwise, they are $(0, \pm c)$ where $c = \sqrt{b^2 - a^2}$.

If $a = b$ then the foci are equal and we get a circle of radius a^2.

The surface of revolution obtained by rotating the ellipse around one of its central axes (x-axis or y-axis, here) is called an *ellipsoid*.

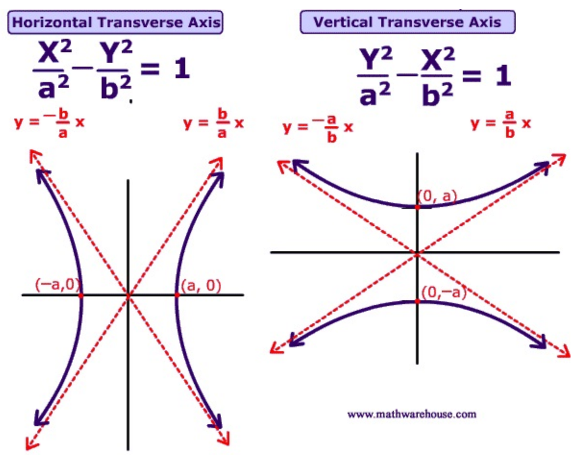
6.1.3 The hyperbola

The hyperbola can be defined geometrically by starting with two foci and now plotting all points for which the *difference* of the distances from the point to the two foci is a constant. (For uses of

[3]Constructing Ellipses: by Steven Dutch, Natural and Applied Sciences, University of Wisconsin - Green Bay, www.uwgb.edu/dutchs/MATHALGO/Ellipses.HTM

[4]Ellipsoid, from http://www.calculatoredge.com/enggcalc/volume.html

hyperbolas in real life, see `http://britton.disted.camosun.bc.ca/jbconics.htm`, for example.)



[5]

One can prove (exercise) that by choosing coordinates appropriately, every hyperbola has the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

for some constants $a, b > 0$. Then the foci are at $(\pm\sqrt{a^2 + b^2}, 0)$. This curve intersects the x-axis where $x^2 = a^2$ or $(\pm a, 0)$; but it does not intersect the y-axis since $y^2 = -b^2$ has no solution. As x goes to $\pm\infty$, note that y must also go to $\pm\infty$ for the equation to hold, and for large x, y,

$$\frac{y^2}{b^2} = 1 + \frac{x^2}{a^2}$$

yields

$$\frac{y^2}{x^2} = \frac{b^2}{x^2} + \frac{b^2}{a^2} \sim \frac{b^2}{a^2}$$

which means asymptotically, the graph approaches the lines $y = \pm\frac{b}{a}x$.

Note that we get a different hyperbola if we swap the roles of x and y, since the equation is not symmetric in the two variables (as we see also from the graph). Rather than swapping the variables, notice instead that you get this transformation by taking the right side to be -1 rather than 1. In fact, the family of curves

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = k$$

gives:

- a hyperbola opening over the x-axis, if $k > 0$
- a pair of intersecting lines through the origin, if $k = 0$
- a hyperbola opening over the y-axis, if $k < 0$.

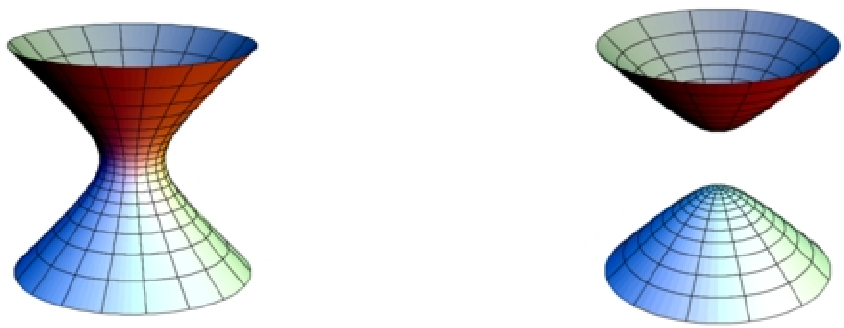
[5]Images of hyperbolas from MathWarehouse.com

Another common equation for a hyperbola is

$$xy = k \quad \Rightarrow \quad y = \frac{k}{x}$$

and this is equivalent to our forms above by a change of variable $\tilde{x} = x + y$, $\tilde{y} = x - y$ (exercise).

This time, there are two possible surfaces of revolution we could obtain, both called *hyperboloids*. If we rotate about one axis, we get a *hyperboloid of one sheet* (looks like the coolant tower of a nuclear reactor) and if we rotate about the other axis, we get the *hyperboloid of two sheets*, which comes in two pieces.



[6]

6.1.4 Exercises

1. Classify all curves of the form $ax^2 + by^2 = c$, according to whether a, b and c are positive, negative or zero. (Since the signs are the important part of the coefficients, it is convenient to write the coefficients as $\pm a^2$ because then you don't need to additionally say whether a is positive or negative.)

2. Identify the following conic sections by type. Sketch the graph, indicating all intercepts and asymptotes (if applicable):

   (a) $3x^2 + 4y^2 = 4$

   (b) $3x^2 - 4y^2 = 3$

   (c) $-3x^2 + 4y^2 = 2$

   (d) $xy = 4$ (Hint: there is a theorem that promises this is one of the conic sections. Sketch the graphe to decide what kind it must be.)

   (e) $x^2 + y^2 + 2x - 4y = -4$

   (f) $x^2 - y^2 + 2x - 2y = 3$

6.2 Quadric surfaces in $\mathbb{R}^3$

One way to derive the equations for the surfaces of revolution defined in the preceding section is as follows. If you are rotating about the x-axis, replace y^2 (the square of the distance of the point

[6]Images of hyperboloids from MathForum.org http://mathforum.org/mathimages/index.php/Hyperboloid

$(x, y) \in \mathbb{R}^2$ to the x-axis) with $y^2 + z^2$ (the square of the distance of the point $(x, y, z) \in \mathbb{R}^3$ to the x-axis). More generally, you can use the equation of an ellipse rather than of a circle in this substitution. In doing so, we come up with equations which are, up to renaming variables, cases of one of the following:

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1$$

or

$$z = \frac{x^2}{a^2} \pm \frac{y^2}{b^2}$$

for constants $a, b, c > 0$.

Actually, you didn't get the minus sign in this last equation from any of the above substitutions; I just threw it in. The collection of these (plus a degenerate version of the first) are called the *quadric surfaces*.

So how do we figure out the shape from the equation? The general strategy, which we'll employ for graphs of two-variable functions later, is the following.

Consider the intersection of the surface with the coordinate planes; this is given by setting one of the variables equal to zero, and so is easy to find. More generally, you can slice the surface parallel to any coordinate plane by setting the corresponding variable (that is, the variable whose axis is perpendicular to that plane) equal to some constant k. These curves are called *traces* of the surface. A typical computer graphing program computes lots of traces and then fills in the surface from there.

Typically, computing the traces relative to one axis is enough to let you visualize the surface; for graphs of functions we'll use the z-axis most often and call the traces *level curves* or *contour lines* and other such. In the following examples, we'll do all the traces to get a feel for the differences.

6.2.1 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$:

Setting $x = 0$ or $y = 0$ or $z = 0$ gives an ellipse. Setting $x = k$ gives

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{k^2}{a^2}$$

which is:

- an ellipse, if $|k| < a$
- the points $(\pm a, 0, 0)$, if $|k| = a$
- empty (no solution) if $|k| > a$.

This describes an ellipsoid. If $a = b = c = r > 0$ then it is a sphere of radius r.

6.2.2 $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$:

Setting $x = 0$ or $y = 0$ gives a hyperbola, but setting $z = 0$ gives an ellipse.

Setting $z = k$ yields

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2}$$

which is an ellipse for any height k, since the right side is always positive. So all the horizontal slices are ellipses.

Setting $x = k$ yields

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 - \frac{k^2}{a^2}$$

which is:

- a hyperbola opening over the y-axis, if $|k| < a$
- the union of the lines $y = \pm\frac{b}{c}z$, if $|k| = a$
- a hyperbola opening over the z-axis, if $|k| > a$.

The result is analogous for the level curves $y = k$. This describes a hyperboloid of one sheet.

6.2.3 $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$:

Setting $x = 0$ gives the empty set; setting $y = 0$ or $z = 0$ gives a hyperbola opening over the x-axis.

Setting $x = k$ yields

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{k^2}{a^2} - 1$$

which is:

- empty, if $|k| < a$
- the points $(\pm a, 0, 0)$ if $|k| = a$
- an ellipse, if $|k| > a$

Setting $y = k$, on the other hand, yields

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 + \frac{k^2}{b^2}$$

which is a hyperbola opening over the z-axis, for any k, since the right side is always positive.

We conclude that this is a hyperboloid of two sheets.

6.2.4 $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$:

This is the surface

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

which is a degenerate form of the two preceding cases. The level curves of z are all ellipses, except when $z = 0$, when it is just a point. By the same reasoning as above, we find that this is a (double) cone (opening over the z-axis).

6.2.5 $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$:

This looks superfically like the equation for the double cone, but note that the z term is not squared.

When $z = 0$, it's the origin. When $x = 0$ or $y = 0$, we have a parabola opening over the z-axis.

Notice that $z \geq 0$. When $z = k > 0$ we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = k$$

which describes an ellipse.

When $x = k$, we get

$$z = \frac{k^2}{a^2} + \frac{y^2}{b^2}$$

which is a parabola opening over the z-axis (shifted vertically by k^2/a^2 units). (Similary for $y = k$.)

So this object is a paraboloid (an *elliptic paraboloid* if $a \neq b$).

6.2.6 $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$:

This is our weirdest shape. It is not a surface of revolution, but captures a bizarre feature of functions of 2 variables that we've never seen in functions of one variable.

When $z = 0$ it is the union of two lines through the origin. For $z = k$, we have various hyperbolas.

When $x = 0$, we have a parabola $z = \frac{-1}{b^2} y^2$, opening downwards on the z-axis. When $x = k \neq 0$, we have the same parabolas, but shifted upwards.

When $y = 0$, we have a parabola $z = \frac{x^2}{a^2}$, opening *upwards* on the z-axis. When $y = k \neq 0$, we have the same parabolas, but shifted downwards.

We call this landscape feature at the origin (that you're at a local minimum in one plane but at a local maximum in another) a *saddle*.

This object is called a *hyperbolic paraboloid*.

6.2.7 Graphs of functions of 2 variables

We are now starting [S, Chapter 11]. From this point onwards, your textbook by Stewart is an excellent reference, with better pictures; which edition you have is irrelevant, as they are all equivalent.

Definition 6.1. A function of two variables is a function whose domain D is contained in $\mathbb{R}^2$ and whose image is contained in $\mathbb{R}$.

We write

$$z = f(x, y)$$

so that x and y are our independent variables and z is now the dependent variable.

Example 6.2. Consider $f(x, y) = x \ln(x^2 - y)$. This is a function of 2 variables and its doamin consists of all (x, y) such that $x^2 - y > 0$ (because otherwise the formula doesn't make sense). So its domain D is a subset of $\mathbb{R}^2$.

Notice how interesting the domain is in this case, as opposed to the one-variable case.

Example 6.3. Consider $f(x, y) = \dfrac{\sqrt{xy - 1}}{x - 1}$. This is a function of 2 variables and its domain consists of all (x, y) such that both the numerator makes sense and the denominator is nonzero, that is,

$$D = \{(x, y) \in \mathbb{R}^2 \mid xy \geq 1; x \neq 1\}.$$

If $x > 0$ then $xy > 1$ is $y > 1/x$; if $x < 0$ then $xy > 1$ is $y < 1/x$. So our domain is the union of these two regions, minus the line $x = 1$ which cuts one of them!

We can similarly define a function of n variables, which we might write

$$f(x_1, \cdots, x_n).$$

Definition 6.4. The *graph* of a function of 2 variables is the set

$$\Gamma = \{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\}.$$

When we draw Γ, we choose the right-handed coordinate system. This is (usually)[7] a surface in $\mathbb{R}^3$.

Notice that the graph of a function of 1 variable was an object in $\mathbb{R}^2$; and the graph of a function of 2 variables is an object in $\mathbb{R}^3$. Similarly, the graph of a real-valued function of n variables is a subset of $\mathbb{R}^{n+1}$.

Example 6.5. $f(x, y) = 2x + y + 1$ has graph $z = 2x + y + 1$ which is the plane

$$2x + y - z = -1$$

which has normal vector $(2, 1, -1)$.

[7] It could happen that you design a ridiculous function whose domain is a line or a point, in which case your graph is a line or a point, lying over that domain.

Example 6.6. $f(x, y) = x^2 + y^2$ has graph $z = x^2 + y^2$, which is a paraboloid.

Example 6.7. $f(x, y) = x^2 - y^2$ has graph $z = x^2 - y^2$, which is a hyperbolic paraboloid (remember, the one with a saddle at the origin).

Example 6.8. $f(x, y) = \sqrt{x^2 + y^2}$ has graph $z = \sqrt{x^2 + y^2}$ which is the upper cone of the double cone $z^2 = x^2 + y^2$.

Example 6.9. $f(x, y) = y^2$ has graph $z = y^2$, which looks like you've slid that parabola up and down the x-axis.

Remark 6.10. Note that the projection of the graph of a function f onto the xy plane is exactly the domain of f. If you use a computer graphics package, you will usually restrict your domain to a square in the xy plane, so that you only see one piece of the infinite surface.

To sketch surfaces, we often draw traces, which are easier to visualize and sketch (because we are so familiar with 2-dimensional graphs). When the surface is the graph of a function, we usually choose

$$z = k$$

which are called the *level curves*, for various values of k.

For example, if f represents altitude, then these level curves are the *contour lines* you see on topographic maps.

If f represents temperature over an area, then the level curves are the *isotherms*.

If f is the Cobb-Douglas production function (determining the total production P of an economy as a function of the amount of labour L (in person-hours/year) and the capital investment K (in dollars))

$$P = bL^{\alpha}K^{1-\alpha}$$

(for $b > 0$, $0 < \alpha < 1$) then the level curves of P represent the various combinations of L and K that yield a given total production P.

Example 6.11. Sketch the level curves of the function $f(x, y) = \ln(x^2 + 4y^2)$.

Solution: Set $k = \ln(x^2 + 4y^2)$. Then

$$x^2 + 4y^2 = e^k$$

which are (wide) ellipses with intercepts $(\pm\sqrt{e^k}, 0)$ and $(0, \pm\frac{1}{2}\sqrt{e^k})$.

We deduce that the graph has a singularity (sometimes called a pole, for obvious reasons) at $(0, 0)$.

Example 6.12. Sketch the level curves of the function $f(x, y) = y \sec(x)$.

Solution: Let's restrict to $-\pi/2 < x < \pi/2$ to fix our ideas. We have

$$k = y \sec(x)$$

or

$$y = k \cos(x)$$

which are obtained by stretching the graph of $y = k\cos(x)$ (on this domain) by k. What's happening at the points where $\cos(x) = 0$ and $\sec(x)$ is undefined? *All* the level curves pass through these points, so the surface we obtain would have a vertical line over those points — hence not be a function. But everywhere in between, we see that the graph is has a lovely linear slope.

6.2.8 Exercises

1. [S]: Appendix B # 9, 47, 49, 51
2. Show that $xy = 1$ defines a hyperbola by making the change of variable $\tilde{x} = x + y$, $\tilde{y} = x - y$ and seeing that with respect to the new variable the relation is in the standard form for a hyperbola. (This is an example of applying the *theory of quadratic forms* (in MAT2141): one diagonalizes the quadratic form xy to put it in standard form.
3. [S]: Section 9.6 #3, 5, 7, 15, 17, 21 (hint: complete the square), 23, 27, 33 (this exercises defines a *ruled surface*).
4. [S]: Section 11.1 # 3 (Cobb-Douglas function), 5, 7, 9, 11, 15, 17, 19, 21, 23 (more interesting), 31-34 (use a computer graphics program), 35-40, 41, 43, 45.

Chapter 7

Calculus on functions of 2 or more variables

7.1 Limits and continuity

The material in this section comes from [S, Chapter 11.2].

We need to upgrade our definitions of convergent sequences (Definition 2.2) and of continuity of a function (Definition 3.6) from $\mathbb{R}$ to $\mathbb{R}^2$.

7.1.1 Sequences in $\mathbb{R}^2$

Write $\|(x,y)\| = \sqrt{x^2+y^2}$ for the length of the vector (x,y). Then the distance between (x,y) and (a,b) is

$$\|(x,y)-(a,b)\| = \|(x-a,y-b)\| = \sqrt{(x-a)^2+(y-b)^2}.$$

Definition 7.1. Given a sequence $\{(x_n,y_n)\}_{n\in\mathbb{N}}$ of vectors in $\mathbb{R}^2$, we say that this sequence *converges to* (a,b) if for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\|(x_n,y_n)-(a,b)\| < \varepsilon.$$

Example 7.2. For example, $\{(\frac{1}{n}, 1+\frac{1}{n})\}_{n\geq 1}$ converges to $(0,1)$, since

$$\|(x_n,y_n)-(a,b)\| = \|(\frac{1}{n},\frac{1}{n}\| = \frac{\sqrt{2}}{n}$$

so if $\varepsilon > 0$, then by choosing $N > \sqrt{2}/\varepsilon$ we has satisfied the conditions of the definition.

Example 7.3. Suppose $\{x_n\}_{n\geq 1} \to a$ in the real numbers. Then

$$\{(x_n,0)\} \to (a,0)$$

and similarly

$$\{(0,x_n)\} \to (0,a).$$

Namely, for any $\varepsilon > 0$ take N such that for all $n \geq N$ we have $|x_n - a| < \varepsilon$. Then

$$\|(x_n, 0) - (a, 0)\| = \sqrt{(x_n - a)^2} = |x_n - a| < \varepsilon$$

as required.

The key thing to remember is that there are many, many more ways to converge to a point in $\mathbb{R}^2$ than there were in $\mathbb{R}$, because you can approach from any direction (not just left and right). For example, all of the following sequences converge to $(0, 0)$:

$$\{(\frac{1}{n}, \frac{1}{n})\}_{n\geq 1}, \{(0, \frac{1}{n})\}_{n\geq 1}, \{(\frac{1}{n}, 0)\}_{n\geq 1}, \{(\frac{47}{n}, \frac{-201}{n})\}_{n\geq 1}, \ldots$$

7.1.2 Limits of functions

Before moving on to continuous functions, let's explore how limits of 2-variable functions behave. In the case of single-variable functions, at a point where the function was undefined you typically saw a vertical asymptote appear (like in $\ln(x)$ at $x = 0$) or that it tried to converge to a point (like $\frac{x^2-4}{x-2}$ at $x = 2$); it was rare to find functions like $\sin(1/x)$ that did something more interesting as $x \to 0$.

Definition 7.4. We say

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for any sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ converging to (a, b) (such that none of its terms are equal to (a, b)), we have

$$\lim_{n\to\infty} f(x_n, y_n) = L.$$

First, an example of where things are as we expect.

Example 7.5. Suppose $f(x, y) = x^2+y^2$. Suppose $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ is a sequence converging to $(0, 0)$. Let us prove that $\{f(x_n, y_n)\}$ converges to $f(0, 0) = 0$, that is, that for any $\varepsilon > 0$ there is some N such that for all $n \geq N$ we ahve $|f(x_n, y_n) - 0| = x_n^2 + y_n^2 < \varepsilon$.

Let $\varepsilon > 0$. Since $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ converges to $(0, 0)$, we know there is some $N \in \mathbb{N}$ such that

$$\|(x_n, y_n) - (0, 0)\| < \sqrt{\varepsilon},$$

which is equivalent, upon squaring, to

$$x_n^2 + y_n^2 < \varepsilon$$

as required.

Now, for some pathological examples.

Example 7.6. Consider the function

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

which is undefined at $(0,0)$. Let's find $\lim_{(x,y)\to(0,0)} f(x,y)$, if it exists.

Suppose we try a sequence like $(x_n, 0)$ with $\{x_n\} \to 0$. Then $f(x_n, 0) = 1$ for every n, which converges to 1.

But if we try a sequence like $(0, y_n)$, with $\{y_n\} \to 0$, then $f(0, y_n) = -1$ for every n, which converges to -1.

Since these are not equal, the limit does not exist. In fact, the graph of the function has a vertical line segment over the origin, between $(0,0,-1)$ and $(0,0,1)$.

If two different sequences give two different answers, then the limit does not exist.

Example 7.7. Consider the function

$$f(x,y) = \frac{xy}{x^2 + y^2}$$

which is undefined at $(0,0)$. Let's find $\lim_{(x,y)\to(0,0)} f(x,y)$, if it exists.

This time, for any sequence of the form

$$\{(x_n, 0)\} \quad \text{or} \quad \{(0, y_n)\}$$

which converges to $(0,0)$, the numerator is 0, so it is again a constant sequence converging to 0.

But now suppose we take a sequence $\{(x_n, x_n)\} \to (0,0)$, that is, a sequence whose points like on the line $y = x$. Then

$$f(x_n, x_n) = \frac{1}{2}$$

for all n, which converges to $\frac{1}{2}$.

Therefore again, the limit does not exist. The graph again contains a vertical line segment at the origin but this time we didn't see it from the x or y directions alone!

To decide if a limit of a 2-variable function exists, it is insufficient to consider the x and y-axes separately.

There are even worse possibilities.

Example 7.8. Consider the function

$$f(x,y) = \frac{x^2 y}{x^4 + y^2}$$

which is undefined at $(0,0)$. Let's find $\lim_{(x,y)\to(0,0)} f(x,y)$, if it exists.

We can approach along any line of the form $y = mx$ by choosing a sequence $\{x_n\} \to 0$ and setting

$$(x_n, y_n) = (x_n, mx_n)$$

which also converges to 0. Then

$$f(x_n, mx_n) = \frac{mx_n^3}{x_n^4 + m^2x_n^2} = \frac{mx_n}{x_n^2 + m^2}$$

which converges to 0 as $n \to \infty$. But even all this is insufficient to prove that the limit exists; for consider instead the sequence

$$(x_n, x_n^2)$$

which gives

$$f(x_n, x_n^2) = \frac{x_n^4}{x_n^4 + x_n^4} = \frac{1}{2}.$$

The limit does not exist here either! See Figure 7.1.

> It is never enough to consider only those sequences that you happen to think of — for the limit to exist, ALL sequences must give the same answer.

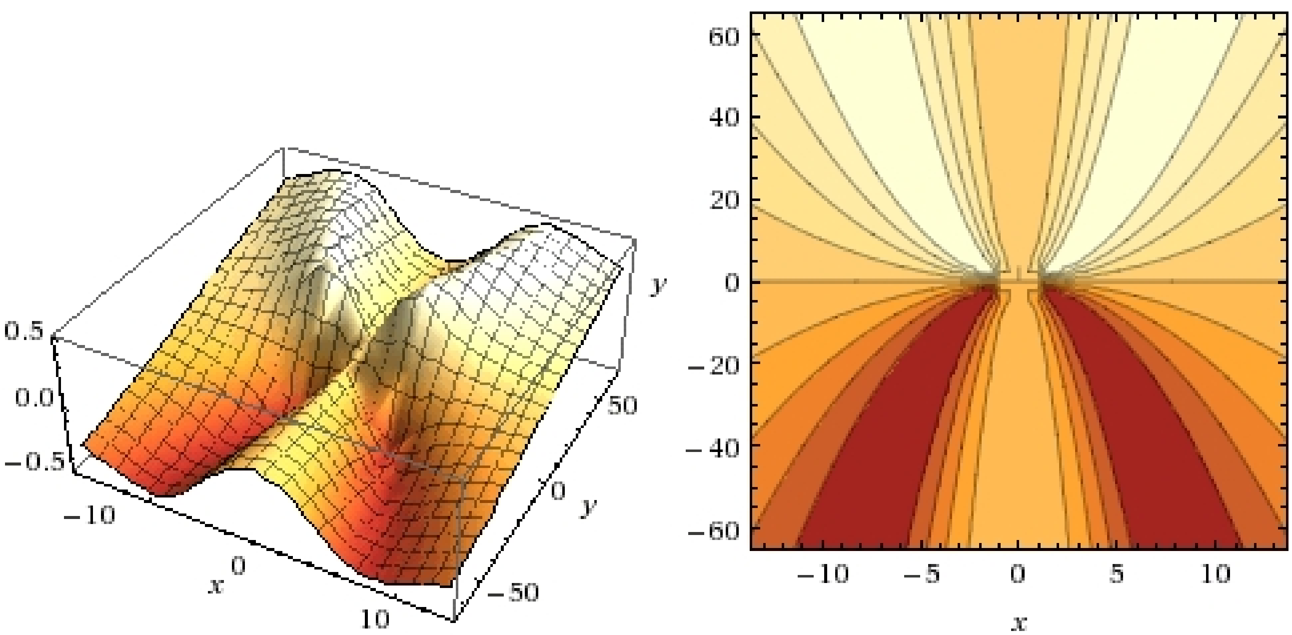


Figure 7.1: The graph and contour lines of the function $f(x, y) = \frac{x^2y}{x^4+y^2}$. The maximum occurs along the curve $y = x^2$ and the minimum along the curve $y = -x^2$. Straight lines to the origin may cross over this ridge but then settle back to 0.

7.1.3 Continuity and composition of functions

Proving that the limit exists and has a given value can be challenging; however, we already have many answers, coming from all of the *continuous* functions.

Definition 7.9. A function f of 2 variables with domain D is continuous at $(a, b) \in D$ if for any sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ of points in D converging to (a, b), we have that

$$\lim_{n\to\infty} f(x_n, y_n) = f(a, b).$$

We say f is continuous if it is continuous at every $(a, b) \in D$.

Example 7.10. The function

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0) \end{cases}$$

is not continuous at $(0, 0)$, because the limit of f as (x, y) goes to $(0, 0)$ does not even exist, let alone equal $f(0, 0)$.

Example 7.11. You can show that (exercise)

$$\lim_{(x,y)\to(a,b)} x = a$$

and

$$\lim_{(x,y)\to(a,b)} y = b.$$

Therefore the functions $f(x, y) = x$ and $f(x, y) = y$ are continuous, as are the constant functions.

It follows from Theorem 2.16 that any function which is a product or sum of continuous functions is continuous, so all polynomial functions of two variables are continuous. Also, any rational function is continuous on its domain.

Remark 7.12. The way to see this is as follows. From Theorem 2.16 we know that if $\{x_n\}_{n\in\mathbb{N}} \to a$ and $\{y_n\}_{n\in\mathbb{N}} \to b$ then for example $\{x_n y_n\}_{n\in\mathbb{N}} \to ab$. Now by the homework in this case $\{(x_n, y_n)\}_{n\in\mathbb{N}} \to (x, y)$. Then if f and g are continuous at (a, b), we know that $\{f(x_n, y_n)\}_{n\in\mathbb{N}} \to f(a, b)$ and $\{g(x_n, y_n)\}_{n\in\mathbb{N}} \to g(a, b)$. Let's set $z_n = f(x_n, y_n)$ and $u_n = g(x_n, y_n)$, remembering that these are real numbers and not vectors. Then $\lim_{n\to\infty} z_n = f(a, b)$ and $\lim_{n\to\infty} u_n = g(a, b)$, so the fact that $\{z_n u_n\}_{n\in\mathbb{N}} \to f(a, b)g(a, b)$ is just saying $\{f(x_n, y_n)g(x_n, y_n)\}_{n\in\mathbb{N}} \to f(a, b)g(a, b)$. That is, the product function fg is also continuous at (x, y).

Example 7.13. The function $\frac{xy+y^2}{1-xy}$ is continuous on its domain $D = \{(x, y) \mid xy \neq 1\}$.

Lemma 7.14. *Suppose f is a function of 2 variables and g is a function of 1 variable such that f is continuous at (a, b), and g is defined and continuous at $f(a, b)$. Then the composition*

$$g \circ f \quad \textit{given by} \quad (g \circ f)(x, y) = g(f(x, y))$$

is a function of 2 variables which is continuous at (a, b).

Example 7.15. Let $f(x, y) = x^2 + y^2 + 1$ and $g(t) = \ln(t)$. The $g \circ f$ is the function

$$g(f(x, y)) = g(x^2 + y^2 + 1) = \ln(x^2 + y^2 + 1)$$

which is by the lemma continuous at every $(x, y) \in \mathbb{R}^2$.

Proof. Let $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ be a sequence converging to (a, b). Since f is continuous at (a, b), we know that the sequence $z_n = f(x_n, y_n)$ converges to $z = f(a, b)$. Since g is continuous at z, we know that

$$\lim_{n\to\infty} g(z_n) = g(z)$$

or equivalently

$$\lim_{n\to\infty} g(f(x_n, y_n)) = g(f(a, b)).$$

□

One can also prove that if h_1, h_2 are functions of 1 variable which are continuous at a and b respectively, and f is a 2-variable function which is defined and continuous at $(h_1(a), h_2(b))$, then $f \circ (h_1, h_2)$, which is the function defined by

$$f \circ (h_1, h_2)(x, y) = f(h_1(x), h_2(y))$$

is continuous at (a, b). (Exercise)

Example 7.16. Let $f(x, y) = x^2 + xy + y^2$, $h_1(t) = \sin(t)$ and $h_2(t) = e^t$. Then

$$f \circ (h_1, h_2)(x, y) = f(h_1(x), h_2(y)) = f(\sin(x), e^y) = \sin^2 x + e^x \sin(x) e^{2y}$$

is continuous at any $(x, y) \in \mathbb{R}^2$.

Remark 7.17. Note that all the definitions in this section generalize easily to functions of 3 or more variables as well!

Conclusions:

- From our bank of known continuous functions of one variable we produce a huge bank of continuous functions of two or more variables, via composition, sums, products and quotients.
- If $f(x, y)$ is piecewise defined, such as

  $$f(x, y) = \begin{cases} \frac{x^2 - y^2}{x - y} & \text{if } x \neq y; \\ 0 & \text{if } x = y \end{cases}$$

  then you need to think. In this example, we see that $f(x, y) = x + y$ when $x \neq y$; since $x + y$ is continuous everywhere, we know that for any point (a, a) on the line $x = y$, the limit of f should be $2a$. But this is equal to $f(a, a)$ only for $a = 0$. Therefore the function is continuous only on the set $\{(x, y) \mid x \neq y\} \cup \{(0, 0)\}$.
- In other cases, like

  $$f(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0); \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

  you find instead that the limit at $(0, 0)$ doesn't even exist, so this function had no hope of being continuous.

7.2 Partial Derivatives

7.2.1 Motivation and setting the stage

So the notion of a continuous function was easy to generalize from 1 variable to 2 variables: we just had to generalize the notion of sequences. But generalizing the notion of a derivative presents new challenges.

Let's think of what we want the derivative to tell us.

Suppose f is the Cobb-Douglas production function (determining the total production of an economy as a function of the amount of labour x (in person-hours/year) and the capital investment y (in dollars))

$$f(x, y) = bx^{\alpha}y^{1-\alpha}$$

(for $b > 0$, $0 < \alpha < 1$). Then the level curves of f ($f(x, y) = k$)) represent the various combinations of x and y that yield a given total production k; see Figure 7.2. I want the derivative to tell me about the rate of change of production; but with respect to what? Well, I could fix y and ask for the rate of change with respect to x; this tells me the estimated effect of increasing the total labour. Or, I could hold x fixed, and work out the rate of change with respect to y; this tells me about the effect of changing the amount of capital. That's a good first goal of a "derivative" and the one we'll tackle here.[1]

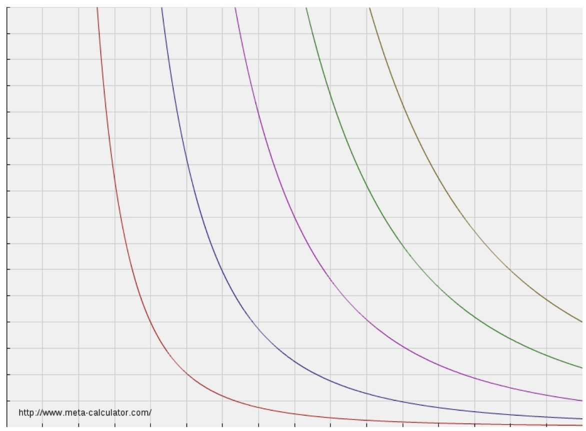


Figure 7.2: Level curves of the Cobb-Douglas production function with x representating labour, y representing capital, and $\alpha = 0.75$.

7.2.2 Defining and computing partial derivatives

Definition 7.18. Let f be a function of two variables defined at a point (x, y). Then the *partial derivative of f with respect to x* is given by

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

(if it exists). Similarly, if the limit

$$\lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

exists, it is called the partial derivative of f with respect to y as is denoted $\frac{\partial f}{\partial y}(x, y)$.

Just like in 1-variable Calculus, the partial derivatives of a 2-variable function can themselves be viewed as functions (of 2-variables!).

[1]But some later questions: if my goal is to increase production f, am I best off increasing x, increasing y, or — evidently most likely and more complicated to answer: increasing both in some optimal ratio? See the chapter on the gradient, coming soon.

Various other notations are also used for the partial derivative:

$$\frac{\partial f}{\partial x} = f_x = D_1(f) = D_x(f) = \cdots$$

and similarly for y.

Example 7.19. Let $f(x, y) = xe^y$. Then

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{(x + h)e^y - xe^y}{h} = \lim_{h \to 0} e^y \frac{(x + h) - x}{h} = e^y$$

since e^y is a term which is unaffected by h, so a constant when you take this limit; and the limit of the quotient is 1. Similarly,

$$\frac{\partial f}{\partial y}(x, y) = \lim_{h \to 0} \frac{xe^{y+h} - xe^y}{h} = \lim_{h \to 0} x \left(\frac{e^{y+h} - e^y}{h} \right) = xe^y$$

since x is independent of h, so a constant insofar as this limit is concerned, and we recognize the limit of that quotient as being the expression for the derivative of the function e^y.

From this example, we realize that the way you take partial derivatives is just to pretend that all other variables are constants.

Rule: to compute f_x, hold all other variables constant.

Example 7.20. Let $f(x, y) = x^2 + 5xy + 3y^2$. Then

$$f_x(x, y) = \frac{\partial f}{\partial x} x^2 + \frac{\partial f}{\partial x} 5xy + \frac{\partial f}{\partial x} 3y^2 = 2x + 5y + 0 = 2x + 5y$$

whereas

$$f_y(x, y) = 0 + 5x + 6y = 5x + 6y.$$

Note that in each case, the graph of the partial derivative function is a plane, just like the graph of the derivative of a quadratic function of one variable is a line.

Example 7.21. Let $f(x, y) = x^y$. When we differentiate with respect to x, then y is held constant. So this function is like x^2 or x^3 and we realize it's the power rule:

$$f_x(x, y) = \frac{\partial f}{\partial x} x^y = yx^{y-1}.$$

On the other hand, to find $f_y(x, y)$, we hold x constant so this function is like 2^x or 3^x. Recall that the derivative of

$$g(x) = 2^x = e^{x \ln(2)}$$

is $g'(x) = \ln(2)e^{x \ln(2)} = \ln(2)2^x$. Therefore

$$f_y(x, y) = \ln(y)x^y.$$

Example 7.22. If $f(x, y) = \sin^2(x/y)$ then we need to apply the chain rule:

$$f_x = 2 \sin(x/y) \cos(x/y) \frac{1}{y}$$

whereas

$$f_y = 2 \sin(x/y) \cos(x/y) \left(\frac{-x}{y^2} \right).$$

Example 7.23. We can also compute $\frac{\partial z}{\partial x}$ via implicit differentiation, if z is defined implicitly. For example, consider the hyperboloid of one sheet

$$x^2 + y^2 - z^2 = 1.$$

Taking partial derivatives of z with respect to x yields

$$2x - 2z\frac{\partial z}{\partial x} = 0$$

or $\frac{\partial z}{\partial x} = x/z$. So at the point $(\sqrt{2}, 0, 1)$, $\frac{\partial z}{\partial x} = 1/\sqrt{2}$ so the value of z is increasing with x. We could also find $\frac{\partial z}{\partial y} = y/z$, so at the point $(\sqrt{2}, 0, 1)$, we see that this derivative is 0. Looking at the graph, we see that this is a point where z goes from increasing as a function of y to decreasing as a function of y.

Another point of view: given $x = f(x, y)$ or else z defined implicitly in terms of x and y, the rules are: $\dfrac{\partial x}{\partial x} = 1$, $\dfrac{\partial y}{\partial x} = 1$, and $\dfrac{\partial z}{\partial x} = z_x = f_x$. Similarly for the other variables.

In the case of an implicitly defined function like the above, you can change your point of view about which variables are independent and which one is dependent. (We could do this with one-variable functions as well, which is how we found the derivatives of some inverse functions.)

Example 7.24. Given

$$x^2 + y^2 - z^2 = 1.$$

we could equally well find $\frac{\partial y}{\partial z}$, for example — the rate of change of y with respect to z, holding x constant. We get

$$2y\frac{\partial y}{\partial z} - 2z = 0$$

so $\frac{\partial y}{\partial z} = z/y$. At $(0, 1, 0)$, we get 0, meaning that that at that point in the trace, we have a critical point (in fact, from the graph we see a minimum, thinking of y as a function of z).

It doesn't matter how many variables we have.

Example 7.25. Suppose $f(x, y, z) = z\sin(xy^2)$. Then

$$f_x = z\cos(xy^2)y^2 = y^2 z\cos(xy^2)$$
$$f_y = z\cos(xy^2)(2xy) = 2xyz\cos(xy^2)$$
$$f_z = \sin(xy^2)$$

7.2.3 Interpretations of partial derivatives

As rates of change: the partial derivative of $z = f(x, y)$ with respect to x is telling us the rate of change of z with respect to x (while y is held constant).

Geometrically: from the definition we see that in fact f_x is telling us about the slope of the trace of the graph given by holding y constant; if $y = b$ for example then the trace is the function

$$g(x) = f(x, b)$$

and $f_x(x, b) = g'(x)$. Similarly for f_y.

Example 7.26. Let $f(x,y) = \sin(x^2 + 2x\sin(y))e^{xy}$. Find $f_x(1,0)$.

Option 1: $f_x(x,y) = \cos(x^2+2x\sin(y))(2x+2\sin(y))e^{xy} + ye^{xy}\sin(x^2+2x\sin(y))$ so $f_x(1,0) = 2\cos(1)$.

Option 2:

$$f_x(1,0) = \frac{d}{dx}\bigg|_{x=1} f(x,0) = \frac{d}{dx}\bigg|_{x=1} \sin(x^2) = 2x\cos(x^2)\bigg|_{x=1} = 2\cos(1).$$

As a matrix: we often store the partials in a matrix called Df. For example, if $f(x,y) = x^2+xy$ then

$$Df(x,y) = \begin{bmatrix} f_x & f_y \end{bmatrix} = \begin{bmatrix} 2x+y & x \end{bmatrix}$$

so that

$$Df(1,2) = \begin{bmatrix} 4 & 1 \end{bmatrix}.$$

We will have use of this matrix shortly.

7.2.4 Higher derivatives

So we can compute partial derivatives and we see their geometric interpretation. (It isn't yet obvious how these partial derivatives are going to be very useful, but we'll get there.)

In one-variable Calculus, you can compute the second derivative $f''(x)$. What is the analogous notion to second derivatives here? It turns out: there are 4. Since f_x is again a two-variable function, we can (and should) differentiate it with respect to x or to y, and similarly for f_y.

Define the *second partial derivatives* as follows (again, with a multitude of possible notation):

$$\begin{aligned}
f_{xx} &= \frac{\partial f_x}{\partial x} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2} \\
f_{xy} &= \frac{\partial f_x}{\partial y} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x} \\
f_{yx} &= \frac{\partial f_y}{\partial x} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y} \\
f_{yy} &= \frac{\partial f_y}{\partial y} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}
\end{aligned}$$

So for example f_{xx} is the usual second derivative of the trace of f (holding y constant), and it tells us about the concavity of this trace. Similarly for f_{yy}. But the mixed partials f_{xy} and f_{yx} are funny: f_{xy} is about the rate of change with respect to y of the slope with respect to x — telling us about how the slope changes as you move transverse to your trace.

Example 7.27. Let $z = x^2 - y^2$, the hyperbolic paraboloid (pringle). We have

$$\begin{aligned}
f_x &= 2x \\
f_y &= -2y
\end{aligned}$$

so that the second partials (= second partial derivatives) are:

$$\begin{aligned} f_{xx} &= 2 \\ f_{xy} &= 0 \\ f_{yx} &= 0 \\ f_{yy} &= -2 \end{aligned}$$

So the first and the last point out that on all traces $y = k$ (respectively, $x = k$), we have a parabola of constant concavity 2 (respectively, -2). The middle ones say: the slope of the trace is constant (=derivative 0) as you change from one trace to another (perpendicular to the trace).

Example 7.28. Let $f(x, y) = 4x^3y + y^2$. Then the partial derivatives are:

$$\begin{aligned} f_x &= 12x^2y \\ f_y &= 4x^3 + 2y \end{aligned}$$

so that the second partials (= second partial derivatives) are:

$$\begin{aligned} f_{xx} &= 24xy \\ f_{xy} &= 12x^2 \\ f_{yx} &= 12x^2 \\ f_{yy} &= 2 \end{aligned}$$

Example 7.29. Let $f(x, y) = x\ln(y)$. Then $f_x = \ln(y)$ and $f_y = \frac{x}{y}$. Therefore

$$\begin{aligned} f_{xx} &= 0 \\ f_{xy} &= \frac{1}{y} \\ f_{yx} &= \frac{1}{y} \\ f_{yy} &= -\frac{x}{y^2} \end{aligned}$$

We've noticed a curious coincidence in these examples.

Theorem 7.30 (Clairaut's Theorem)**.** *Suppose f is defined on an open disk D containing the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D then*

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Note: we need to require the open disk so that we can work out the derivatives from all directions (as opposed to if (a, b) is stuck at the boundary of the domain). The hypotheses are a little stronger than needed but you do need continuity of the mixed partials at (a, b) (see exercises).

Proof. (See [S, Appendix E].) First, a plausibility argument:

Let's consider the approximation to $f_{xy}(a, b)$ given by

$$\frac{f_x(a, b + h) - f_x(a, b)}{h}$$

which we could further approximate by

$$\frac{f(a+h,b+h)-f(a,b+h)-(f(a+h,b)-f(a,b))}{h^2}.$$

We see right away that this is equal to

$$\frac{f(a+h,b+h)-f(a+h,b)-(f(a,b+h)-f(a,b))}{h^2} \sim \frac{f_y(a+h,b)-f_y(a,b)}{h}.$$

To turn this into a proof, we want to go backwards from our estimates; we'll use the Mean Value Theorem. Let

$$g(x) = f(x,b+h) - f(x,b);$$

so this is a differentiable function of 1 variable with derivative $g'(x) = f_x(x,b+h) - f_x(x,b)$. By the MVT, there is some c between a and $a+h$ such that

$$\frac{g(a+h)-g(a)}{h} = g'(c) = f_x(c,b+h) - f_x(c,b)$$

In other words,

$$\frac{f(a+h,b+h)-f(a,b+h)-(f(a+h,b)-f(a,b))}{h^2} = \frac{f_x(c,b+h)-f_x(c,b)}{h}$$

Now $f_x(c,y)$ is a differentiable function of 1 variable, with derivative $f_{xy}(c,y)$, so applying the MVT again gives us a d between b and $b+h$ such that

$$\frac{f_x(c,b+h)-f_x(c,b)}{h} = f_{xy}(c,d).$$

So for each $h > 0$ we have found a point (c,d) near (a,b) (exercise: how near?) for which our quotient equals $f_{xy}(c,d)$.

Doing the same process with the roles of x and y reversed gives us a point (u,v) near (a,b) for which the quotient equals $f_{yx}(u,v)$.

Now choose a sequence $h_n \to 0$, and for each n construct the corresponding sequences (c_n,d_n) and (u_n,v_n); by construction, each of these converge to (a,b). Since f_{xy} and f_{yx} are continuous at (a,b), as $n\to\infty$ these converge to $f_{xy}(a,b)$ and $f_{yx}(a,b)$, respectively. But both are equal to

$$\lim_{n\to\infty} \frac{f(a+h_n,b+h_n)-f(a,b+h_n)-(f(a+h_n,b)-f(a,b))}{h_n^2}$$

and so are equal to each other. □

Remark 7.31. So the idea of the proof was: show that the difference quotient is equal to $f_{xy}(c,d)$ for some point (c,d) near (a,b) and that it is also equal to $f_{yx}(u,v)$ for some point (u,v) near (a,b). "Near" means so that the x coordinate is between a and $a+h$ and the y-coordinate is between b and $b+h$. (In class, we used t for h in the x-component.) As you make $h\to 0$, these points limit to (a,b), and since the mixed partials are continuous, their values converge to $f_{xy}(a,b)$ and $f_{yx}(a,b)$, respectively. But they must be equal, since at every h, they were the same value.

7.3 Differentiability

In one-variable Calculus, a function is said to be differentiable if its derivative exists. Let's take a look at an example to see that the existence of partials is not sufficient to imply differentiability (whatever "differentiability" means!).

Example 7.32. Consider

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We have seen previously that this function is not continuous at $(0,0)$ because if you approach $(0,0)$ along any non-horizontal, non-vertical line your limit is not 0.

However, if you look at the trace $y = 0$, you see that it's the constant zero function of x, and so $f_x(0,0) = \frac{d}{dx}|_{x=0} f(x,0) = \frac{d}{dx}|_{x=0} 0 = 0$, no problem — this partial derivative exists.

Same for $f_y(0,0)$, since the trace $x = 0$ is again the constant function of y.

Therefore both partial derivatives exist. But it does not seem reasonable to say that the function is differentiable at $(0,0)$, because it isn't even continuous there! Our intuitive notion of "differentiability" is that the function looks locally very linear around that point, but there's no way to argue this is true of this example at $(0,0)$.

What's going on in this example? The same kind of thing as we saw with limits: we're not seeing enough of the function when we approach along a straight line to make any judgement calls. What's a good fix?

Example 7.33. Let f be as in the previous example and look at the whole partial derivative as a function, not just at the point:

$$f_x(x,y) = \frac{y(x^2+y^2) - 2x^2y}{(x^2+y^2)^2} = \frac{y(y^2-x^2)}{(x^2+y^2)^2}$$

Look along the trace $x = 0$ to see that f_x, as a function, does not have a limit as $(x,y) \to (0,0)$ (even though it looks benign if you approach along the trace $y = 0$, as we noted in our calculation above).

From this example, we draw some conclusions, which we'll talk about in detail now:

- Differentiability should be about the existence of a tangent plane. So what does that mean?
- If the partial derivatives are continuous functions, then that is enough to make the function differentiable. (But a function can be differentiable even without continuous partials; obscure examples to be found in MAT2122.)

7.3.1 The tangent plane

In Calculus of 1 variable, we had a notion of the tangent line to the curve at a point. To write down the equation of the tangent line, we needed its slope, which is the derivative at that point, as well as the coordinates of the point in question.

The equation of the tangent line to $y = f(x)$ through a point (x_0, y_0) is

$$y - y_0 = f'(x_0)(x - x_0).$$

This was found using the point-slope formula for the equation of a line.

Example 7.34. Find the equation of the tangent line to $y = x^3$ at $x = 1$.

Solution: The equation for the tangent line at the point (x_0, y_0) is $y - y_0 = f'(x_0)(x - x_0)$, yielding in this case $y - 1 = 3(x - 1)$ or $y = 3x - 2$.

What is the analogue for a function of 2-variables? It will be a plane. Let us derive the correct expression for what the equation of the tangent plane should be.

The normal equation for a plane is

$$ax + by + cz = d$$

where (a, b, c) is a normal vector to the plane. In our case, since we're taking the tangent plane to the graph of a function, we expect our plane to be a function of x and y also. So we can rewrite

$$z = Ax + By + C.$$

Now our plane will pass through a point (x_0, y_0, z_0) so we can eliminate the constant be writing this as

$$z - z_0 = A(x - x_0) + B(y - y_0).$$

Now what are A and B? Notice that if we take the trace $y = y_0$ we get

$$z - z_0 = A(x - x_0)$$

and this line is exactly the tangent line to the trace $y = y_0$, so $A = f_x(x_0, y_0)$. Similarly, taking trace $x = x_0$ yields $z - z_0 = B(y - y_0)$ whence we deduce that $B = f_y(x_0, y_0)$.

Excellent. We've just worked out:

Lemma 7.35. *If the graph of $z = f(x, y)$ has a tangent plane at the point (x_0, y_0, z_0), and if the partials of f exist there, then the tangent plane is given by the equation*

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

This leaves the question of existence open. We'll state a theorem shortly that promises the tangent plane exists if the partials are continuous functions; let's assume that is true for now.

Example 7.36. Find the equation of the tangent plane to $z = 3x^2 + y^2$ at $(x_0, y_0, z_0) = (1, 1, 4)$.

We have $Df(x, y) = \begin{bmatrix} f_x & f_y \end{bmatrix} = \begin{bmatrix} 6x & 2y \end{bmatrix}$ so the partials are continuous. We have $Df(1, 1) = \begin{bmatrix} 6 & 2 \end{bmatrix}$. Therefore the equation for the tangent plane is

$$z - 4 = 6(x - 1) + 2(y - 1).$$

7.3.2 Linear approximation

As shown in [S], if you zoom in on the graph of $z = f(x, y)$ at (x_0, y_0) and the graph of the tangent plane, the two become indistinguishable if you get sufficiently close. This is analogous to how we thought of tangent lines to curves : that they offer a linear approximation that is quite accurate if you are close enough. We can say this for functions of 2 variables as well.

Suppose f is a function of 2 variables. The tangent plane to the graph of f at (x_0, y_0, z_0) is itself the graph of a function, namely, the function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

(where we have replaced z with $L(x, y)$ to give a name to this function and have noticed that $z_0 = f(x_0, y_0)$).

Remark 7.37. Sometimes, to make this more readable, we'll denote the point by (a, b) and then the function is

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

We call L the *linearization* of f at (x_0, y_0) (or (a, b)). The values $L(x, y)$ provide a *linear approximation* to $f(x, y)$ for (x, y) close to (x_0, y_0).

Example 7.38. Given $f(x, y) = 3x^2 + y^2$ and its linearization $L(x, y) = 4 + 6(x - 1) + 2(y - 1)$ at $(a, b) = (1, 1)$, compare values of f and L near $(1, 1)$.

$f(1, 2) = 7$ whereas $L(1, 2) = 8$.

$f(1.1, 1) = 4.63$ whereas $L(1.1, 1) = 4.6$.

So it's a decent approximation.

Another way to write the linearization is in matrix form:

$$L(x, y) = f(x_0, y_0) + Df(x_0, y_0) \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

so that you see that $Df(x_0, y_0)$ is a linear transformation such that

$$Df(x_0, y_0) \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

is a measure of the increase or decrease of f in the direction of the vector $(x - x_0, y - y_0)$.

7.3.3 Differentiability of f

We finally get to the definition of the differentiability of f. We saw that if f has partials at (x_0, y_0) and its tangent plane at the point (x_0, y_0, z_0), if it exists, is given by the equation

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

We also saw that if the graph of $z = f(x, y)$ has a tangent plane at the point (x_0, y_0, z_0), then the linearization (which is just the function whose graph is the alleged tangent plane)

$$L(x, y) = f(x_0, y_0) + Df(x_0, y_0) \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

is a "good approximation" to f at (x_0, y_0).

The definition of differentiability of f comes down to saying in a precise way what "good approximation" means.

Definition 7.39. Let f be a function of two variables and (a, b) a point in its domain where the partials exist. Let $L(x, y)$ be the linearization of f at (a, b). Write $\Delta x = x - a$ and $\Delta y = y - b$. Then f is *differentiable* at (a, b) if

$$\lim_{(x,y)\to(a,b)} \frac{f(x, y) - L(x, y)}{\sqrt{(x - a)^2 + (y - b)^2}} = 0.$$

Remark 7.40. Compare this with the 1-variable definition. There, $L(x) = f(a) + f'(a)(x - a)$ so

$$\lim_{x\to a} \frac{f(x) - L(x)}{x - a} = \lim_{x\to a} \frac{f(x) - f(a) - f'(a)(x - a)}{x - a} = \lim_{x\to a} \left(\frac{f(x) - f(a)}{x - a} - f'(a) \right) = 0.$$

Thus this is consistent.

Remark 7.41. In MAT2122, you will go on to prove that in fact if f is differentiable, then $L(x, y)$ is the *unique* linear function which satisfies this property, so that the existence of the partials follows from differentiability.

The definition is quite complex; what it can be used for is to show that f can be approximated by a linear function in a very specific way. It can also be used to prove the following useful theorem.

Theorem 7.42. *If the partials f_x and f_y are continuous at (a, b), then f is differentiable at (a, b).*

You can see the proof in MAT2122, or in [S, Appendix D].

Example 7.43. The function $f(x, y) = x^2 e^{xy}$ is differentiable because its partial derivatives

$$f_x = 2xe^{xy} + ye^{xy}$$

and

$$f_y = x^3 e^{xy}$$

are continuous everywhere.

Example 7.44. The function $f(x, y) = x^{1/3}$ has

$$f_x = \frac{1}{3x^{2/3}}, \quad f_y = 0,$$

so its partials exist and are continuous on the domain $\{(x, y) \mid x \neq 0\}$; hence f is differentiable there. But along $x = 0$, we see from the graph of f that the tangent plane is the xz-plane, which is not the graph of any linear function L. We deduce that f is not differentiable there.

7.3.4 The chain rule

In Calculus of 1 variable, the chain rule was stated as

$$\frac{d}{dx} f \circ g = \frac{df}{dg}\frac{dg}{dx}$$

or

$$(f(g(x)))' = f'(g(x))g'(x).$$

To generalize this to several variables, let's first look at some different kinds of composition.

- $f(t)$ composed with $h(x)$: $f(h(x))$ — this is the usual one-variable case
- $f(t)$ composed with $h(x, y)$: $f(h(x, y))$ — so $f \circ h$ is a function of 2 variables
- $f(t)$ composed with $h(x, y, z)$: $f(h(x, y, z))$ — so $f \circ h$ is a function of 3 variables
- (etc)
- $f(x, y)$ composed with $\vec{h}(t) = (h_1(t), h_2(t))$: $f(h_1(t), h_2(t))$ — so $f \circ \vec{h}$ is a function of t alone
- $f(x, y)$ composed with $\vec{h}(x, y) = (h_1(x, y), h_2(x, y))$: $f(h_1(x, y), h_2(x, y))$ — so $f \circ \vec{h}$ is a function of 2 variables
- $f(x, y)$ composed with $\vec{h}(x, y, z) = (h_1(x, y, z), h_2(x, y, z))$: $f(h_1(x, y, z), h_2(x, y, z))$ — so $f \circ \vec{h}$ is a function of 3 variables
- (etc)

To differentiate, we just need to keep track of all the different ways in which the composition depends on each of its variables, and add them.

Suppose $z = f(x, y)$ and $x = h_1(t)$ and $y = h_2(t)$. So z is a function of t alone, and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Example 7.45. Say $z = x^2y$ and $(x, y) = (\sin(t), \cos(t))$. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = 2xy \cdot \cos(t) + x^2 \cdot (-\sin(t))$$

which is correct, in the sense that x and y are functions of t, so the whole thing is a function of t; but we could also simplify by plugging in the values of x and y to get

$$\frac{dz}{dt} = 2\sin(t)\cos^2(t) - \sin^3(t).$$

We compare with the direct calculation: $f(\sin(t), \cos(t)) = \sin^2(t)\cos(t)$, whose derivative is exactly the above.

Suppose $z = f(u, v)$ and $(u, v) = (h_1(x, y), h_2(x, y))$. So z is a function of x and y, and we have

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x},$$

and

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.$$

Example 7.46. Suppose $z = uv$ and $u = \sin(x)$ and $v = x - y$. Then

$$\frac{\partial z}{\partial x} = v \cdot \cos(x) + u \cdot 1 = (x - y)\cos(x) + \sin(x)$$

and

$$\frac{\partial z}{\partial y} = v \cdot 0 + u\dot{(} - 1) = -\sin(x).$$

We compare by direct calculation:

$$z = uv = (x - y)\sin(x) \quad Dz = \begin{bmatrix}\sin(x) + (x - y)\cos(x) & -\sin(x)\end{bmatrix}$$

as expected.

These rules generalize to any number of variables. You can often write it as a tree diagram where each branch corresponds to a summand of the final formula.

The general rule can be written as:

Suppose $z = f(x_1, \cdots, x_n)$ is differentiable, and each x_i is a function of the variables $u_1, u_2, \cdots, u_m$. Then for each $i \in \{1, \cdots, m\}$ we have

$$\frac{\partial z}{\partial u_i} = \sum_{j=1}^{n} \frac{\partial z}{\partial x_j}\frac{\partial x_j}{\partial u_i}$$

There is another way to keep track of the chain rule, one which is a bit more natural and makes use of the fact that the composition of linear transformations is just a matrix product (and the derivative is really a linear transformation of the tangent plane (if you put the origin at the point of contact)).

7.3.5 The derivative in matrix form (optional)

We can write the derivative in matrix form.

Definition 7.47. If f is a differentiable function from $\mathbb{R}^n$ to $\mathbb{R}$, on variables $x_1, \cdots, x_n$, then the derivative of f is the matrix

$$Df = \begin{bmatrix}\frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n}\end{bmatrix}$$

Example 7.48. Write $x_1 = x$, $x_2 = y$. If $f(x, y) = x^2 + 4xy$ then $Df(x, y) = \begin{bmatrix}2x + 4y & 4x\end{bmatrix}$.

Example 7.49. If $f(x_1, x_2, x_3) = x_1^2 + x_2 x_3$ then $Df(x_1, x_2, x_3) = \begin{bmatrix} 2x_1 & x_3 & x_2 \end{bmatrix}$.

Example 7.50. If $f(x_1) = \sin(x_1)$ then $Df(x_1) = \begin{bmatrix} \cos(x_1) \end{bmatrix}$. That's OK. In linear algebra, we learn that we usually identify a 1×1 matrix with a scalar, and thus we recover our "usual" derivative of a function of one variable.

Now what about vector-valued functions, that is, ones whose range lies in $\mathbb{R}^2$ or $\mathbb{R}^3$, for example? We start with the simplest case.

Definition 7.51. Suppose $\vec{h}$ is a differentiable[2] function from $\mathbb{R}$ to $\mathbb{R}^m$, written $\vec{h}(t) = (h_1(t), h_2(t), \cdots, h_m(t))$, or, in vector form,

$$\vec{h}(t) = \begin{bmatrix} h_1(t) \\ h_2(t) \\ \vdots \\ h_m(t) \end{bmatrix}.$$

Then the derivative of $\vec{h}$ is the matrix

$$D\vec{h} = \begin{bmatrix} \frac{d\,h_1}{dt} \\ \frac{d\,h_2}{dt} \\ \vdots \\ \frac{d\,h_m}{dt} \end{bmatrix}.$$

Example 7.52. Let $\vec{h}(t) = (\cos(t), \sin(t))$. The image of this function in $\mathbb{R}^2$ is the circle of radius 1 (but if you were to graph it on axes t, x, y you'd see its spiral over time).

We have

$$D\vec{h}(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}.$$

Plotting this derivative vector on the same graph as the function reveals that it does indeed give a tangent vector to the curve (of length related to the speed at which you are tracing the curve).

Example 7.53. Let $\vec{h}(t) = (t, t, t)$. This is the straight line $t(1, 1, 1)$ in $\mathbb{R}^3$. Its derivative is

$$D\vec{h}(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

as expected.

The general case is a combination of these two.

Definition 7.54. Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is a differentiable vector-valued function of n variables, written as

$$f(x_1, \cdots, x_n) = (f_1, f_2, \cdots, f_m)$$

[2] We say a vector-valued function is differentiable if each of its component functions (here $h_1(t)$, $h_2(t)$, etc) are differentiable.

where each $f_i = f_i(x_1, \cdots, x_n)$ is a function of n variables. Then the derivative of f is

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Example 7.55. Let $f(x, y) = (x^2 + y, \sin(xy))$. Then

$$Df(x, y) = \begin{bmatrix} 2x & 1 \\ y\cos(xy) & x\cos(xy) \end{bmatrix}$$

We note that this matrix is not symmetric in general! Its size tells you the dimension of the domain and the range of your function.

7.3.6 The chain rule: matrix form (optional)

Theorem 7.56. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ and $g\colon \mathbb{R}^p \to \mathbb{R}^n$ be differentiable functions, and let $\vec{a}$ be in the domain of g. Then*

$$D(f \circ g)(\vec{a}) = (Df)(g(\vec{a}))Dg(\vec{a})$$

where the right side is matrix multiplication.

We won't prove the theorem in this course; but let's do some examples.

Example 7.57. If $n = m = p = 1$ then $Df = f'$ and $Dg = g'$ and $\vec{a} = a$ so this is the usual chain rule of one variable.

Example 7.58. Suppose $f(x, y) = e^{xy}$ and $\vec{h}(t) = (t^2, \sin(t))$. Then

$$Df = \begin{bmatrix} ye^x & xe^y \end{bmatrix} \quad \Rightarrow \quad (Df)(\vec{h}(t)) = Df(t^2, \sin(t)) = \begin{bmatrix} e^{t^2}\sin(t) & t^2 e^{\sin(t)} \end{bmatrix},$$

and

$$D\vec{h}(t) = \begin{bmatrix} 2t \\ \cos(t) \end{bmatrix}$$

So for any t we have

$$\begin{aligned} D(f \circ \vec{h})(t) &= (Df)(\vec{h}(t))D\vec{h}(t) \\ &= \begin{bmatrix} e^{t^2}\sin(t) & t^2 e^{\sin(t)} \end{bmatrix} \begin{bmatrix} 2t \\ \cos(t) \end{bmatrix} \\ &= 2te^{t^2}\sin(t) + t^2\cos(t)e^{\sin(t)}. \end{aligned}$$

To compare, we could also compute directly:

$$f(\vec{h}(t)) = f(t^2, \sin(t)) = e^{t^2 \sin(t)}$$

whose derivative is exactly what we've computed.

Example 7.59. Suppose $f(x, y) = (f_1, f_2, f_3)$, $g(t) = (g_1, g_2)$ and $h(u, v)$ are given and we want to compute the derivative of their composition at $(1, 2)$. Say $h(1, 2) = 3$ and $g(3) = (4, 2)$. Suppose we have found

$$Df(4,2) = \begin{bmatrix} 3 & 4 \\ -1 & 0 \\ 1 & 1 \end{bmatrix},$$

$$Dg(3) = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

$$Dh(1,2) = \begin{bmatrix} 4 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} D(f \circ g \circ h)(1,2) &= Df(g(h(1,2))) \cdot Dg(h(1,2)) \cdot Dh(1,2) \\ &= Df(4,2)Dg(3)Dh(1,2) \\ &= \begin{bmatrix} 3 & 4 \\ -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 4 \\ -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 4 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 0 \\ 4 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

where the zeros in the last column imply that f does not vary as a function of v. This makes sense, because h doesn't vary as a function of v, and that was the only route for v to possibly affect f.

7.4 Directional derivatives

From now on, let us assume that f is a differentiable function of 2 variables. The material in this section is from [S, 11.6].

The linearization of f at (a, b) gave us a means to answer a question that has been plaguing us since the start: if the partials tell us how f changes as we go along the graph of f in the x (respectively, y) direction, how do we find the rate of change of f in an arbitrary direction?

Recall that a *unit vector* is a vector of length 1. If $u = (u_1, u_2)$ is a vector, then

$$\frac{1}{\|u\|}(u_1, u_2) = \left(\frac{u_1}{\|u\|}, \frac{u_2}{\|u\|} \right)$$

is a unit vector.

Example 7.60. Let $u = (1, 3)$. Then $\|u\| = \sqrt{1+9} = \sqrt{10}$, so it is not a unit vector. But let $v = \frac{1}{\sqrt{10}}(1, 3)$; then

$$\|v\| = \sqrt{(\frac{1}{\sqrt{10}})^2 + (\frac{3}{\sqrt{10}})^2} = \sqrt{\frac{1}{10} + \frac{9}{10}} = 1$$

so v is indeed a unit vector.

We think of unit vectors as encoding a direction but being neutral on magnitude.

Definition 7.61. The directional derivative of f at the point (x, y) in the direction of the unit vector $u = (a, b)$ is given by

$$D_u f(x, y) = \lim_{h \to 0} \frac{f(x + ha, y + hb) - f(x, y)}{h}$$

if this limit exists.

Note that this coincides for $u = (1, 0)$ with f_x and for $u = (0, 1)$ with f_y.

Theorem 7.62. *Let f be differentiable. Then*

$$D_u f = Df \cdot u$$

where this is the matrix product

$$D_u f = \begin{bmatrix} f_x & f_y \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = f_x(x, y)a + f_y(x, y)b.$$

Proof. Let $g(h) = f(x + ha, y + hb) = f \circ s(h)$, where $s(h) = (x + ha, y + hb)$. Then looking at the definition we deduce that $D_u f(x, y) = g'(0)$. By the chain rule,

$$g'(0) = \frac{\partial f}{\partial x}\frac{\partial s_1}{\partial h} + \frac{\partial f}{\partial y}\frac{\partial s_2}{\partial h} = f_x(s(0))a + f_y(s(0))b = f_x(x, y)a + f_y(x, y)b$$

as required. □

So computing the directional derivative just amounts to multiplying your derivative matrix by the vector giving the direction.

Example 7.63. Find the directional derivative of $f(x, y) = x^2 + y^2$ in the direction of the vector $(1, 3)$ at the point $(4, -1)$.

By a preceding example, a unit vector in this direction is $u = (\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}})$.

The derivative matrix is

$$Df = \begin{bmatrix} 2x & 2y \end{bmatrix}$$

So

$$D_u f(4, -1) = \begin{bmatrix} 8 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix} = \frac{2}{\sqrt{10}}$$

which says that the function is increasing in that direction from that point.

Example 7.64. Find the directional derivative of $f(x, y) = x^2 + y^2$ in the direction that makes a $\pi/3$ angle with the positive x-axis, at any point (a, b).

Solution: now $u = (\cos(\pi/3), \sin(\pi/3)) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, since in particular a unit vector is a point on the unit circle.

So

$$D_u f(a, b) = \begin{bmatrix} 2a & 2b \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = a + \sqrt{3}b.$$

7.4.1 The gradient and maximum rate of change

But what is particularly interesting to answer is the question: in what direction is the function increasing the fastest, from a given point?

Definition 7.65. Let f be a differentiable function and define at each point (x, y) the vector

$$\vec{\nabla} f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y});$$

this is just the transpose of the matrix Df. This vector is called the *gradient vector* of f at (x, y).

Example 7.66. Find the gradient of $f(x, y) = 3x^2 + 4xy + e^x$ at $(x, y) = (0, 3)$.

Solution: $\vec{\nabla} f = (f_x, f_y) = (6x + 4y + e^x, 4x)$ so $\vec{\nabla} f(0, 3) = (13, 0)$.

The directional derivative and the gradient are related by the formula $$D_u f(x, y) = \vec{\nabla} f(x, y) \cdot u$$ where the product on the right is the dot product of vectors.

So the gradient is just another way to talk about the derivative; but it has another important interpretation.

Lemma 7.67. *Let f be a differentiable function. Then at every point (x, y), $\vec{\nabla} f(x, y)$ points in the direction of maximum increase of the function.*

Proof. The increase of the function in the direction of the unit vector u is given by

$$D_u f = \vec{\nabla} f \cdot u$$

From linear algebra, we know we can rewrite this as

$$D_u f = \|\vec{\nabla} f\| \|u\| \cos(\theta) = \|\vec{\nabla} f\| \cos(\theta)$$

where θ is the angle between u and $\vec{\nabla} f$. Since $\|u\| = 1$, and $\|\vec{\nabla} f\|$ is a constant, we see this is maximized when $\theta = 0$. That is, the direction in which $D_U f$ is maximal is the same as the direction of $\vec{\nabla} f$. □

Example 7.68. Consider the Cobb-Douglas production function

$$z = x^{0.75} y^{0.25}$$

where x represents total labour and y represents capital; see Figure 7.2. Suppose that the economy is currently in state $(1, 16)$. What ratio of labour capital investment will give the largest boost to production?

Solution: Our function is $f(x, y) = x^{3/4} y^{1/4}$ so

$$\vec{\nabla} f(x, y) = \left(\frac{3y^{1/4}}{4x^{1/4}}, \frac{x^{3/4}}{4y^{3/4}} \right).$$

At the point $(1, 16)$, we have $\vec{\nabla} f(1, 16) = \left(\frac{3}{2}, 2\right)$, so we should increase labour and capital in a $1.5 : 2$ ratio for maximum benefit.

In contrast, you get the same production at the point $(16, \frac{1}{256})$ (that is, this point is on the same level curve); at that point, however, your gradient vector is $\vec{\nabla} f(16, \frac{1}{256}) = \left(\frac{3}{32}, 128\right)$, which corresponds to almost purely increasing capital (y).

Note, however, that $\vec{\nabla} f$ is usually not a unit vector! So if you want to compute the *maximum directional derivative of* f, you'd first set

$$u = \frac{1}{\|\vec{\nabla} f\|} \vec{\nabla} f$$

so that

$$D_u(f) = \vec{\nabla} f \cdot \left(\frac{1}{\|\vec{\nabla} f\|} \vec{\nabla} f \right) = \frac{1}{\|\vec{\nabla} f\|} \vec{\nabla} f \cdot \vec{\nabla} f = \|\vec{\nabla} f\|$$

since $u \cdot u = \|u\|^2$.

The maximum rate of change of f at (x, y) is $\|\vec{\nabla} f(x, y)\|$.

Example 7.69. What is the slope of steepest ascent from the point $(1, \pi)$ of the hilly region whose topography is defined by $z = \sin(xy)$?

Solution: We compute the gradient $\vec{\nabla} f(x, y) = (y\cos(xy), x\cos(xy))$; so at the point $(1, 0)$ this yields $\vec{\nabla} f(1, \pi) = (-\pi, -1)$.

Therefore since $\|(-\pi, -1)\| = \sqrt{\pi^2 + 1} \sim 3.3$, the maximum slope is 3.3 and it is attained by heading in the direction $(-\pi, -1)$.

In contrast, at $(1, 0)$ we get $\vec{\nabla} f(1, 0) = (0, 1)$ which gives maximum slope 1 in the direction of the x-axis. See Figure 7.3 for some insight.

So this is quite handy: by computing the gradient you know in which direction you should go to increase your function value the most. Similarly, the negative of the gradient vector is the direction to go to decrease your function value most (exercise).

7.4.2 Gradients vs level curves and level surfaces

So we get the maximum value $\|\vec{\nabla} f\|$ (and minimum value $-\|\vec{\nabla} f\|$) of the directional derivative when we are in the same (and opposite) direction of the gradient vector.

So in what direction do we get zero? Recall that $\vec{\nabla} f \cdot u = 0$ if and only if u is orthogonal (perpendicular) to $\vec{\nabla} f$. Also, $D_u(f) = 0$ means that the value of the function does not change in that direction — in other words, it must coincide with the direction of the level curve at that point!

At every differentiable point, the gradient vector is orthogonal to the level curve.

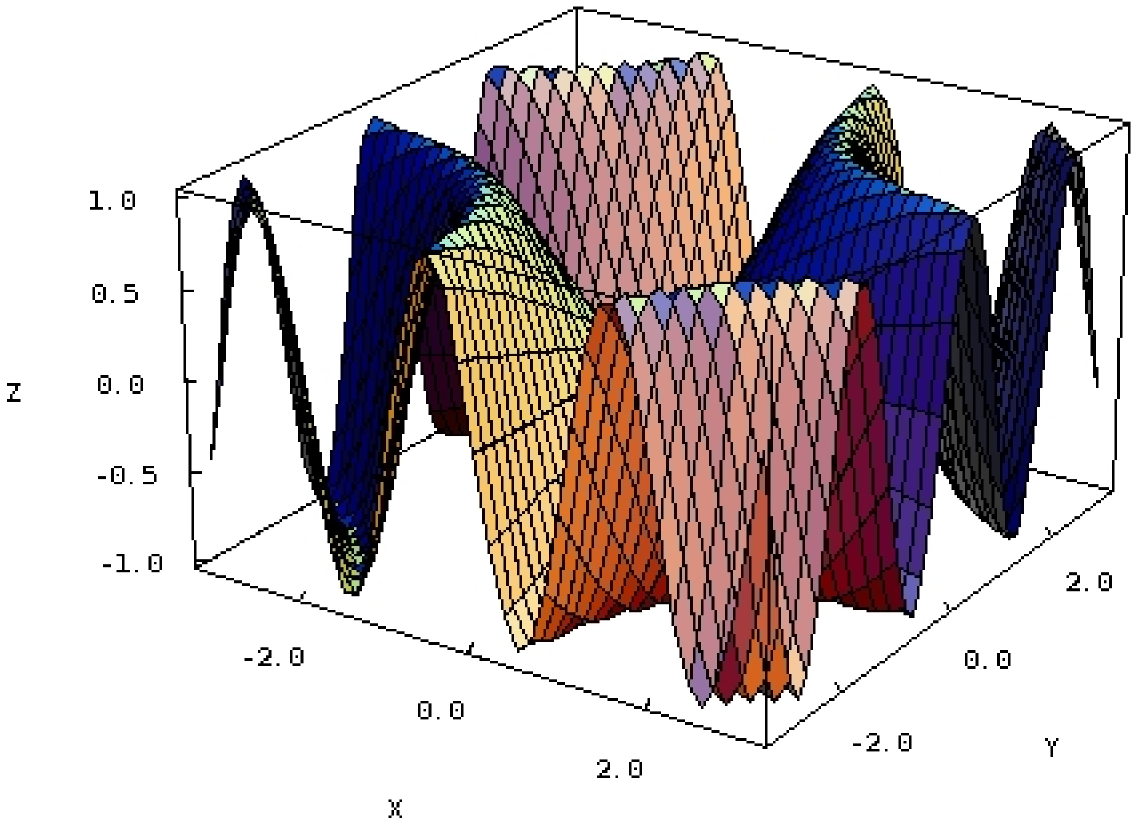


Figure 7.3: A portion of the graph of $z = \sin(xy)$. Note that the oscillations become steeper and more frequent as you head out from the origin.

Example 7.70. Consider $z = x^2 + y^2$ (the paraboloid). The level curves are circles, for $z > 0$.

The gradient vector at each (x, y) is $\vec{\nabla} f = (2x, 2y)$. Let's sketch the gradient vector on top of the contour plot, drawing longer arrows for larger vectors and smaller arrows for smaller vectors, and keeping the direction correct.

What we see (in Figure 7.4) is that at each point, the gradient vector is orthogonal to the level curve and is pointing in the direction of greatest increase of the function. We also see that the function is getting steeper as we go away from the origin (something we would also see if we drew the contour lines/level curves properly).

This is nice, but we appreciate this interpretation better for functions of 3 variables. There, the level sets are not curves, but surfaces, and so the gradient is a vector which is orthogonal to a level surface — in other words, it is a normal vector to the tangent plane of the surface at that point.

Example 7.71. Let $F(x, y, z) = x^2 + 2y^2 + 3z^2$. The graph of F is a 4-dimensional paraboloid, whose level surfaces are the ellipsoids

$$x^2 + 2y^2 + 3z^2 = k$$

for varying $z = k$. So $\vec{\nabla} F = (2x, 4y, 6z)$ is orthogonal to this surface at the point (x, y, z). Thus

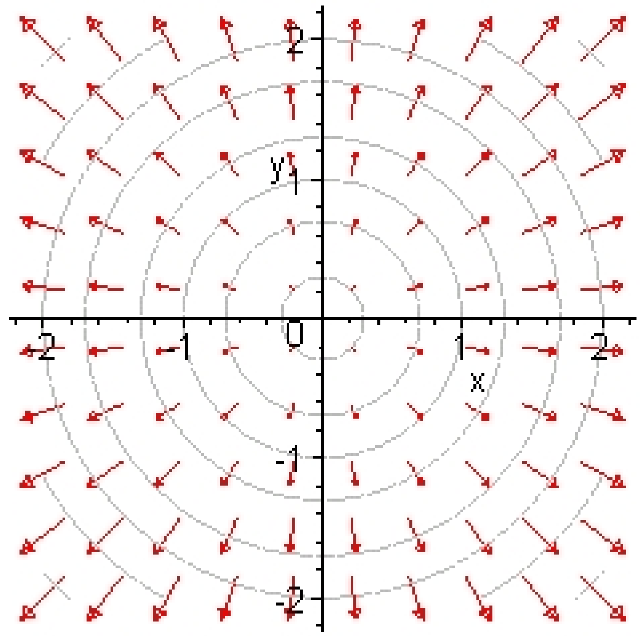


Figure 7.4: Some level curves of $z = x^2 + y^2$ together with arrows representing the gradient at many points.

a normal vector for the tangent plane to the level surface at the point $(-1, 2, 7)$ (which lies on the level surface for $k = 156$) is

$$\vec{\nabla} F = (-2, 8, 42)$$

giving equation of tangent plane

$$-2(x + 1) + 8(y - 2) + 42(x - 7) = 0.$$

7.5 Exercises

1. Let $f(x, y) = x$ and $g(x, y) = y$. Prove that

$$\lim_{(x,y)\to(a,b)} f(x, y) = a$$

and

$$\lim_{(x,y)\to(a,b)} g(x, y) = b,$$

by showing that if $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ is any sequence converging to (a, b), then the sequence $\{f(x_n, y_n)\}_{n\in\mathbb{N}}$ converges to a and the sequence $\{g(x_n, y_n)\}_{n\in\mathbb{N}}$ converges to b.

2. Prove that if h_1, h_2 are functions of 1 variable which are continuous at a and b respectively, and f is a 2-variable function which is defined and continuous at $(h_1(a), h_2(b))$, then $f \circ (h_1, h_2)$, which is the function defined by

$$f \circ (h_1, h_2)(x, y) = f(h_1(x), h_2(y))$$

is continuous at (a, b).

3. Stewart, Chapter 11.2, #5, 7, 9, 11, 13, 15, 17, 19, 23, 27, 29
4. Stewart, Chapter 11.2 #31 (note that this is a function of 3 variables, meaning its domain is a subset of $\mathbb{R}^3$ (and its graph would be an object in $\mathbb{R}^4$, which we won't attempt to draw!).
5. Stewart, Chapter 11.3 #11, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 45, 47, 49 (this one is more interesting), 51, 55, 57, 59, 63, 91 (interesting)
6. (Done in class also) Show that

$$\lim_{h\to 0}\lim_{t\to 0} h^t \neq \lim_{t\to 0}\lim_{h\to 0} h^t$$

   and use this to explain why simply writing down the definitions of $f_{xy}(a,b)$ and $f_{yx}(a,b)$ is insufficient to conclude equality.
7. An alternate definition of differentiability is that there exist functions e_1 and e_2 such that

$$f(x,y) = L(x,y) + e_1(x,y)\Delta + e_2(x,y)\Delta y$$

   where $\lim_{(x,y)\to(a,b)} e_1(x,y) = 0$ and $\lim_{(x,y)\to(a,b)} e_2(x,y) = 0$. Show that this is equivalent to our definition.
8. Rewrite the linearization of f using the directional derivative. To do this best, evaluate L on a point $(x_0, y_0) + tu$ where u is a unit vector. Alternately, the norm of $(x - x_0, y - y_0)$ will come into your formula.
9. Stewart, Section 11.4 #1,3,5, 9, 11, 13 , 15, 17, 19, 21 (good one); we didn't do differentials or parametric surfaces, which are the rest of the exercises.
10. Stewart Section 11.5 #1–12, 13–16 (very good), 17–19, 21–25
11. Prove that the direction u in which $D_u f$ is minimized is the same as the direction of $-\vec{\nabla} f$.
12. Stewart, Section 11.6 #1, 5, 7, 9, 11, 13, 17, 21, 23, 25, 27, 35, 39–44 (very good), 47, 49, 51, 53, 55

Chapter 8

Random extras.

8.1 Parametric curves in $\mathbb{R}^2$

(See [S, Section 1.7].)

In 2 dimensions, the solution set to any equation in the variables x and y is either empty, or a "curve" in the plane. For example

$$y = f(x)$$

describes the curve which is the graph of the function f, whereas

$$y^2 = x(x^2 - 1)$$

describes an "elliptic curve,"[1], which is not the graph of a function, since it doesn't pass the vertical line test.

Another way to describe these curves is by defining a new parameter t (often thought of as time) and defining a new function

$$\mathbf{r}(t) = (x(t), y(t))$$

where as t runs over its domain, these points $(x(t), y(t))$ trace out the curve. For example

$$\{(t, f(t)) \mid t \in \mathbb{R}\}$$

is the *parametric form* of the graph of $y = f(x)$, whereas

$$\{(t^2, t^3) \mid t \in \mathbb{R}\}$$

is the *parametric form* of the curve $x^3 = y^2$. We can't parametrize $y^2 = x(x^2 - 1)$ with continuous functions since the graph is not connected.

The parametrization of a curve is not unique! For example,

$$\{(\cos(t), \sin(t)) \mid t \in \mathbb{R}\}$$

[1] which has nothing to do with an ellipse, despite the suggestive name

describes the circle of radius 1; but so does

$$\{(\cos(2t), \sin(2t)) \mid t \in \mathbb{R}\}$$

(going at twice the speed) and

$$\{(\cos(t), \sin(t)) \mid t \in [0, 2\pi)\}$$

(going around only once).

The *derivative* of $\mathbf{r}(t)$ is just

$$\mathbf{r}'(t) = \left(x'(t), y'(t)\right).$$

You can check (exercise) that if this is the graph of a function, then $\mathbf{r}'(t)$ is just the tangent vector to the curve in our usual sense. In fact (thinking of $\mathbf{r}(t)$ as describing the trajectory of a particle over time) note that $x'(t)$ is the rate of change of the x-coordinate of the particle with respect to time t; and similarly for $y(t)$, so the vector $\mathbf{r}'(t)$ encodes the resolution into coordinates of its velocity.

From linear algebra, you know that lines can be described parametrically; as an exercise, show that a line can be written in the form

$$\mathbf{r}(t) = \mathbf{r}(0) + t\mathbf{r}'(0)$$

which is a nice analogy with what you'd write in the 1-variable case.

For a curve to be "smooth", we require that these derivatives are both continuous; but additionally (as you can see for the parametric curve (t^2, t^3)) we cannot have both derivatives vanishing at the same time, that is, we should never have $x'(t) = 0 = y'(t)$. At such a point, your particle has come to a full stop, and so *could* dramatically change direction, causing a kink (cusp) in the curve.

Remark 8.1. The arc length of a smooth parametric curve $(x(t), y(t))$ from $t = a$ to $t = b$ is calculated by

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

.

1. Show that if $\mathbf{r}(t) = (t, f(t))$ for a differentiable function f, then the derivative $\mathbf{r}'(t)$ is a tangent vector to the curve at time t.

2. Recall from linear algebra that a line in $\mathbb{R}^n$ can be written in parametric form as

$$\mathbf{r}(t) = \mathbf{p} + t\mathbf{v}$$

   for $t \in \mathbb{R}$. Show that $\mathbf{p} = \mathbf{r}(0)$ and $\mathbf{v} = \mathbf{r}'(0)$.

3. [S]: Section 1.7 #22, 23, 26

4. [S]: Section 10.1 #1, 5, 7, 19–24

5. [S]: Section 6.4 # 5, 15, 32, 33

8.2 Curves in $\mathbb{R}^3$

(See [S, Section 10.1].)

Now, in 3 dimensions, if you consider the solution set to an equation in the variables x, y and z, the solution is not a curve but rather a "surface" in $\mathbb{R}^3$. For example

$$x^2 + y^2 + z^2 = 1$$

describes a sphere of radius 1. The heuristic (meaning, it's usually true) is: in 3-space one has 3 degress of freedom; each independent equation cuts down one degree of freedom, and thus gives you a 2-dimensional object. You have learned this philosophy in Linear Algebra, for example, where it is defined and proven precisely.

So how would you describe a curve in $\mathbb{R}^3$? One option: as the intersection of two surfaces, like you can describe a line as the intersection of two planes. That is, the curve is the solution set to a *pair* of equations in 3 variables.

The usual option: as a parametric curve. For example,

$$(\cos(t), \sin(t), -2)$$

describes a circle of radius 1 parallel to the xy plane at height $z = -2$.

So a parametric curve in $\mathbb{R}^3$ is the image of a function

$$\mathbf{r}(t) = (x(t), y(t), z(t))$$

which we could call a *vector-valued function*; in fact the derivative

$$\mathbf{r}'(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

is the vector representing the tangent to the curve at a given point. The physical interpretation: if a particle constrained to follow the curve were released at time t, it would fly off in direction $(x'(t), y'(t), z'(t))$ at speed equal to the magnitude of this vector.

You will study *vector-valued functions*, like parametric curves and vector fields, in more detail in MAT2122.

Index

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