Concordia University

Department of mechanical and industrial engineering

ENGR 233, applied advanced calculus

Name: **Student ID:**

Winter 2009 Second midterm time: 75 minutes

Answer all the questions in the space provided

Question (1) (3 marks)

Find the curl and the divergence of the given vector field

F (x, y, z) = (4xy) i + ($2x^2$ + 2yz) j + ($3z^2$ + y^2) k

Solution

$$\text{curl } \mathbf{F} = 0; \ \ \text{div } \mathbf{F} = 4y + 8z$$

Question (2) (3 marks)

Determine whether the given vector field is a gradient field, if so find a potential function Φ for F.

F (x, y, z) = (x^3 + y) i + (x + y^3) j

Solution

$P_y = 1 = Q_x$ and the vector field is a gradient field. $\phi_x = x^3 + y$, $\phi = \frac{1}{4}x^4 + xy + g(y)$, $\phi_y = x + g'(y) = x + y^3$,

$g(y) - \frac{1}{4}y^4$, $\phi = \frac{1}{4}x^4 + xy + \frac{1}{4}y^4$

Question (3) (3 marks)

Show that the given integral is independent of the path, evaluate

$$\int_{(-2,3,1)}^{(0,0,0)} 2xzdx + 2yzdy + (x^2 + y^2)dz$$

Solution

$P_y = 0 = Q_x$, $Q_z = 2y = R_y$, $R_x = 2x = P_z$ and the integral is independent of path. Parameterize the line segment between the points by $x = -2(1-t)$, $y = 3(1-t)$, $z = 1-t$, for $0 \le t \le 1$. Then $dx = 2\,dt$, $dy = -3\,dt$, $dz = -dt$, and

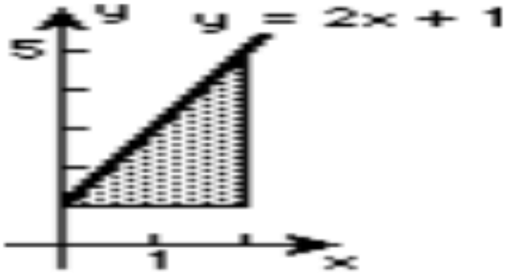
$$\begin{aligned}
&\int_{(-2,3,1)}^{(0,0,0)} 2xz\,dx + 2yz\,dy + (x^2 + y^2)\,dz \\
&= \int_0^1 [-4(1-t)^2(2) + 6(1-t)^2(-3) + 4(1-t)^2(-1) + 9(1-t)^2(-1)]\,dt \\
&= \int_0^1 -39(1-t)^2\,dt = 13(1-t)^3 \Big|_0^1 = -13.
\end{aligned}$$

Question (4) (3 marks)

Sketch the region of integration for the given iterated integral

$$\int_0^2 \int_1^{2x+1} F(x, y)dydx$$

Solution



Question (5) (3 marks)

Evaluate the given iterated integral by reversing the order of integration

$$\int_0^1 \int_x^1 \frac{1}{1+y^4}\, dy dx$$

Solution

$$\int_0^1 \int_x^1 \frac{1}{1+y^4}\, dy\, dx = \int_0^1 \int_0^y \frac{1}{1+y^4}\, dx\, dy = \int_0^1 \frac{x}{1+y^4}\Big|_0^y dy = \int_0^1 \frac{y}{1+y^4}\, dy$$

$$= \frac{1}{2}\tan^{-1} y^2 \Big|_0^1 = \frac{\pi}{8}$$

Question (6) (3 marks)

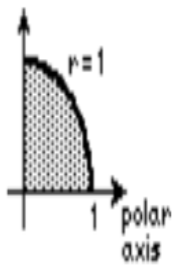
Evaluate the given iterated integral by changing the polar coordinates

$$\int_0^1 \int_0^{\sqrt{1-y^2}} e^{x^2+y^2}\, dxdy$$

Solution

$$\int_0^1\int_0^{\sqrt{1-y^2}} e^{x^2+y^2}\,dx\,dy = \int_0^{\pi/2}\int_0^1 e^{r^2} r\,dr\,d\theta = \int_0^{\pi/2} \frac{1}{2}e^{r^2}\Big|_0^1\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}(e-1)\,d\theta = \frac{\pi(e-1)}{4}$$



Question (7) (3 marks)

Use green's theorem to evaluate the given line integral

$$\oint_c (x + y^2)dx + (2x^2 - y)dy$$

Where c is the boundary of the region determined by the graphs of

$Y = x^2$ and $y = 4$

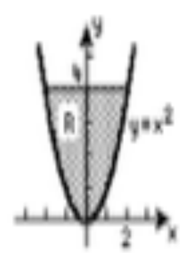
Solution

$$P = x + y^2,\ P_y = 2y,\ Q = 2x^2 - y,\ Q_x = 4x$$

$$\oint_C (x+y^2)\,dx + (2x^2 - y)\,dy = \iint_R (4x - 2y)\,dA = \int_{-2}^{2}\int_{x^2}^{4}(4x - 2y)\,dy\,dx$$

$$= \int_{-2}^{2}(4xy - y^2)\Big|_{x^2}^{4}\,dx = \int_{-2}^{2}(16x - 16 - 4x^3 + x^4)\,dx$$

$$= \left(8x^2 - 16x - x^4 + \frac{1}{5}x^5\right)\Big|_{-2}^{2} = -\frac{96}{5}$$



Question (8) (3 marks)

Use a double integral in polar coordinates to find the area of the region bounded by the graph of the given polar equation r = 2 + cos Θ

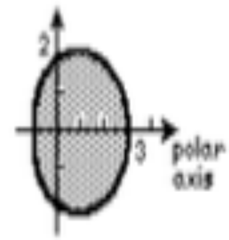
Solution

Using symmetry,

$$A = 2\int_0^{\pi}\int_0^{2+\cos\theta} r\,dr\,d\theta = 2\int_0^{\pi} \frac{1}{2}r^2\Big|_0^{2+\cos\theta} d\theta = \int_0^{\pi}(2+\cos\theta)^2\,d\theta$$

$$= \int_0^{\pi}(4+4\cos\theta+\cos^2\theta)\,d\theta = \left(4\theta+4\sin\theta+\frac{1}{2}\theta+\frac{1}{4}\cos 2\theta\right)\Big|_0^{\pi}$$

$$= \left(4\pi+\frac{\pi}{2}+\frac{1}{4}\right)-\left(\frac{1}{4}\right) = \frac{9\pi}{2}.$$



Question (9) (3 marks)

Evaluate the given partial integral

$$\int_{\sqrt{y}}^{1} y \ln x dx$$

Solution

$$\int_{\sqrt{y}}^{1} y \ln x\,dx \qquad \boxed{\text{Integration by parts}}$$

$$= y(x\ln x - x)\Big|_{\sqrt{y}}^{1} = y(0-1) - y(\sqrt{y}\ln\sqrt{y} - \sqrt{y}) = -y - y\sqrt{y}\left(\frac{1}{2}\ln y - 1\right)$$

Question (10) (3 marks)

Evaluate $\oint_c G(x,y)dx$, $\oint_c G(x,y)dy$ and $\oint_c G(x,y)ds$ on the indicated curve C.

G (x, y) = $x^3 + 2xy^2 + 2x$, $x = 2t$, $y = t^2$, $0 \leq t \leq 1$

Solution

$$\int_C (x^3 + 2xy^2 + 2x)\,dx = \int_0^1 [8t^3 + 2(2t)(t^4) + 2(2t)]2\,dt = 2\int_0^1 (8t^3 + 4t^5 + 4t)\,dt$$

$$= 2\left(2t^4 + \frac{2}{3}t^6 + 2t^2\right)\Big|_0^1 = \frac{28}{3}$$

$$\int_C (x^3 + 2xy^2 + 2x)\,dy = \int_0^1 [8t^3 + 2(2t)(t^4) + 2(2t)]2t\,dt = 2\int_0^1 (8t^4 + 4t^6 + 4t^2)\,dt$$

$$= 2\left(\frac{8}{5}t^5 + \frac{4}{7}t^7 + \frac{4}{3}t^3\right)\Big|_0^1 = \frac{736}{105}$$

$$\int_C (x^3 + 2xy^2 + 2x)\,ds = \int_0^1 [8t^3 + 2(2t)(t^4) + 2(2t)]\sqrt{4 + 4t^2}\,dt = 8\int_0^1 t(1 + t^2)^{5/2}\,dt$$

$$= 8\left(\frac{1}{7}(1 + t^2)^{7/2}\right)\Big|_0^1 = \frac{8}{7}(2^{7/2} - 1)$$