Economics 325/2 A: Fall 2013 Midterm 2

Instructions: You have 45 minutes. There are a total of 100 marks. Use your time wisely.

1. Consider the function

$$f(x) = 8x^{\frac{3}{4}} - \frac{4}{5}x^{\frac{5}{4}} \text{ for } x > 0.$$

i) [**20 marks**] Find the first-order condition and solve. Determine if your solution yields a global minimum or maximum. ii) [**7 marks**] Find the global maximum of

$$g(x) = \left(\left(e^{2x^{\frac{3}{4}}}\right)^{-2}\left(e^{x^{\frac{5}{4}}}\right)^{\frac{2}{5}}\right)^{-2} \text{ for } x > 0.$$

iii) [**8 marks**] Find the elasticity of

$$h(x) = \frac{\left(\left(\frac{3}{2}\right)^{\frac{3}{5}}\left(e^{2\ln(x)}x^2\right)^{\frac{1}{10}} + x^0 - x^3\left(\frac{1}{e^{2\ln(x)}}\right)^{\frac{3}{2}}\right)^{\frac{5}{3}}\left(\ln\left(\exp\left(x^{\frac{1}{10}}\right)\right)\right)^2}{\sqrt{x^{\frac{2}{5}}}} \text{ for } x > 0.$$

2. i) [**20 marks**] Consider the matrices

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Calculate AA^T, $\det\left[AA^T\right]$ and $\left(AA^T\right)^{-1}$. Calculate $\det[B]$ using the Laplace expansion along the first row. Calculate $\det\left[B^{-1}\right]$, $\det\left[B^T\right]$, and $\det\left[\left(B^TB\left(B^TB\right)^{-1}B^TB\right)^{-1}\right]$. ii) [**15 marks**] Prove that the inverse of *any* symmetric matrix is symmetric (assuming the inverse exists). If C and D are any two symmetric $n \times n$ matrices, prove that $E = (CD + DC)^{-1}$ is symmetric (assuming the inverse exists.) iii) [**10 marks**] Suppose P is an $n \times n$ symmetric idempotent matrix, and z is an $n \times 1$ vector. Show that $x = Pz$ and $y = (I - P)z$ are orthogonal.

3. Suppose a household has a utility function U and budget constraint as

$$U = \ln\left(e^{2\ln(Q_2)}\ln\left(e^{Q_1}\right)\right) + 2\ln\left(\sqrt{e^{\ln(Q_2)}}\right), \; Y = P_1Q_1 + P_2Q_2.$$

i) [**15 Marks**] Simplify the utility function. Write U as a function of Q_1 only, find the first-order condition for utility maximization and solve for the optimal Q_1 and Q_2. ii) [**5 Marks**] Verify the second-order condition for a global utility maximum.

Economics 325/2 A: Fall 2013 Midterm 2 Answers

1. Consider the function

$$f(x) = 8x^{\frac{3}{4}} - \frac{4}{5}x^{\frac{5}{4}} \text{ for } x > 0.$$

i) [**20 marks**] Find the first-order condition and solve. Determine if your solution yields a global minimum or maximum. ii) [**7 marks**] Find the global maximum of

$$g(x) = \left(\left(e^{2x^{\frac{3}{4}}} \right)^{-2} \left(e^{x^{\frac{5}{4}}} \right)^{\frac{2}{5}} \right)^{-2} \text{ for } x > 0.$$

iii) [**8 marks**] Find the elasticity of

$$h(x) = \frac{\left(\left(\frac{3}{2}\right)^{\frac{3}{5}} \left(e^{2\ln(x)}x^2\right)^{\frac{1}{10}} + x^0 - x^3 \left(\frac{1}{e^{2\ln(x)}}\right)^{\frac{3}{2}} \right)^{\frac{5}{3}} \left(\ln\left(\exp\left(x^{\frac{1}{10}} \right) \right) \right)^2}{\sqrt{x^{\frac{2}{5}}}} \text{ for } x > 0.$$

Answer: i) [**20 marks**] We have

$$f'(x) = 6x^{-\frac{1}{4}} - x^{\frac{1}{4}}$$

and so the first-order condition $f'(x^*) = 0$ is

$$6(x^*)^{-\frac{1}{4}} - (x^*)^{\frac{1}{4}} = 0.$$

[**Marker: For full marks** the student must put a $*$ on x as x^*, and set the derivative equal to 0. Failing to set the derivative equal to 0 is a more serious mistake and hence should have a greater penalty. **Here and for every question on every test.**] Solving we have

$$6(x^*)^{-\frac{1}{4}} = (x^*)^{\frac{1}{4}} \Longrightarrow (x^*)^{\frac{1}{4}}(x^*)^{\frac{1}{4}} = 6 \Longrightarrow (x^*)^{\frac{1}{2}} = 6 \Longrightarrow x^* = 6^2 = 36.$$

As well

$$f''(x) = -\frac{6}{4}\underbrace{x^{-\frac{5}{4}}}_{+} - \frac{1}{4}\underbrace{x^{-\frac{3}{4}}}_{+} < 0$$

for **all** x and so $f(x)$ is **globally** concave. [**Marker: For full marks** the student must state 'for **all** x' and indicate that $x > 0 \Longrightarrow x^a > 0$; for example by writing $\underbrace{x^{-\frac{3}{4}}}_{+}$. **Here and for every question and on every test.**] Since $f(x)$ is globally concave it follows that $x^* = 36$ yields a global maximum.

ii) **[7 marks]** We have

$$
\begin{aligned}
g(x) &= \left(\left(e^{2x^{\frac{3}{4}}}\right)^{-2}\left(e^{x^{\frac{5}{4}}}\right)^{\frac{2}{5}}\right)^{-2} = \left(e^{2x^{\frac{3}{4}}}\right)^{4}\left(e^{x^{\frac{5}{4}}}\right)^{-\frac{4}{5}} \\
&= e^{8x^{\frac{3}{4}}}e^{-\frac{4}{5}x^{\frac{5}{4}}} = e^{8x^{\frac{3}{4}}-\frac{4}{5}x^{\frac{5}{4}}} = e^{f(x)}
\end{aligned}
$$

so that $g(x)$ is a monotonic transformation of $f(x) = 8x^{\frac{3}{4}} - \frac{4}{5}x^{\frac{5}{4}}$ (since $\frac{d}{dx}e^x = e^x > 0$) and so it follows that $f(x)$ and $g(x)$ have the same global maximum at $x^* = 36$.

iii) **[8 marks]** We have

$$
\begin{aligned}
h(x) &= \frac{\left(\left(\frac{3}{2}\right)^{\frac{3}{5}}\left(e^{2\ln(x)}x^2\right)^{\frac{1}{10}} + x^0 - x^3\left(\frac{1}{e^{2\ln(x)}}\right)^{\frac{3}{2}}\right)^{\frac{5}{3}}\left(\ln\left(\exp\left(x^{\frac{1}{10}}\right)\right)\right)^2}{\sqrt{x^{\frac{2}{5}}}} \\
&= \frac{\left(\left(\frac{3}{2}\right)^{\frac{3}{5}}\left(x^2x^2\right)^{\frac{1}{10}} + 1 - x^3\left(\frac{1}{x^2}\right)^{\frac{3}{2}}\right)^{\frac{5}{3}}\left(x^{\frac{1}{10}}\right)^2}{x^{\frac{1}{5}}} \\
&= \frac{\left(\left(\frac{3}{2}\right)^{\frac{3}{5}}x^{\frac{2}{5}} + 1 - x^3\left(x^{-2}\right)^{\frac{3}{2}}\right)^{\frac{5}{3}}x^{\frac{1}{5}}}{x^{\frac{1}{5}}} \\
&= \left(\left(\frac{3}{2}\right)^{\frac{3}{5}}x^{\frac{2}{5}} + 1 - x^3x^{-3}\right)^{\frac{5}{3}} \\
&= \left(\left(\frac{3}{2}\right)^{\frac{3}{5}}x^{\frac{2}{5}}\right)^{\frac{5}{3}} = \frac{3}{2}x^{\frac{2}{3}}.
\end{aligned}
$$

Since $h(x)$ is of the form Ax^b, the elasticity is the exponent $\frac{2}{3}$.

2. i) **[20 marks]** Consider the matrices

$$
A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}.
$$

Calculate AA^T, $\det\left[AA^T\right]$ and $\left(AA^T\right)^{-1}$. Calculate $\det[B]$ using the Laplace expansion along the first row. Calculate $\det\left[B^{-1}\right]$, $\det\left[B^T\right]$, and $\det\left[\left(B^TB\left(B^TB\right)^{-1}B^TB\right)^{-1}\right]$.

ii) **[15 marks]** Prove that the inverse of *any* symmetric matrix is symmetric (assuming the inverse exists). If C and D are any two symmetric $n \times n$ matrices, prove that $E = (CD + DC)^{-1}$ is symmetric (assuming the inverse exists.) iii) **[10 marks]** Suppose

P is an $n \times n$ symmetric idempotent matrix, and z is an $n \times 1$ vector. Show that $x = Pz$ and $y = (I - P)\, z$ are orthogonal.

Answer: i) **[20 marks]** We have

$$\begin{aligned} AA^T &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \\ \det\left[AA^T\right] &= \det \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} = 3 \times 14 - 6 \times 6 = 6 \\ \left(AA^T\right)^{-1} &= \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}^{-1} = \frac{1}{6} \begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix} = \begin{bmatrix} \frac{7}{3} & -1 \\ -1 & \frac{1}{2} \end{bmatrix}. \end{aligned}$$

We have

$$\det[B] = \det \begin{bmatrix} 4 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 4 \times \det \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} - 2 \times \det \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + 1 \times \det \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = 2.$$

Using the properties of determinants (the easy way) we have

$$\det\left[B^{-1}\right] = \frac{1}{\det[B]} = \frac{1}{2} \text{ and } \det\left[B^T\right] = \det[B] = 2.$$

As well

$$\begin{aligned} \det\left[\left(B^T B \underbrace{\left(B^T B\right)^{-1} B^T B}_{I}\right)^{-1}\right] &= \det\left[\left(B^T B\right)^{-1}\right] = \frac{1}{\det\left[B^T B\right]} = \frac{1}{\det\left[B^T\right] \det[B]} \\ &= \frac{1}{\det[B] \det[B]} = \frac{1}{2 \times 2} = \frac{1}{4}. \end{aligned}$$

ii) **[15 marks]** If C is symmetric then $C^T = C$. For $D \equiv C^{-1}$ we then have $D = D^T$ as

$$D^T = \left(C^{-1}\right)^T = \left(C^T\right)^{-1} = C^{-1} = D.$$

For $E = (CD + DC)^{-1}$ we have $C^T = C$ and $D^T = D$ leads to $E^T = E$ as

$$\begin{aligned} E^T &= \left((CD + DC)^{-1}\right)^T = \left((CD + DC)^T\right)^{-1} \\ &= \left((CD)^T + (DC)^T\right)^{-1} \\ &= \left(D^T C^T + C^T D^T\right)^{-1} \\ &= (DC + CD)^{-1} \\ &= (CD + DC)^{-1} = E. \end{aligned}$$

iii) **[10 marks]** Since P is idempotent we have $PP = P$. Since P is symmetric we have $P^T = P$. Now $x = Pz$ and $y = (I - P)z$ are orthogonal or $x^T y = 0$ since

$$\begin{aligned} x^T y &= (Pz)^T ((I-P)z) = z^T P^T (I-P)z \\ &= z^T P (I-P) z = z^T (P - PP) z = z^T (P-P) z = z^T 0 z = 0. \end{aligned}$$

3. Suppose a household has a utility function U and budget constraint as

$$U = \ln\left(e^{2\ln(Q_2)} \ln\left(e^{Q_1}\right)\right) + 2\ln\left(\sqrt{e^{\ln(Q_2)}}\right), \; Y = P_1 Q_1 + P_2 Q_2.$$

i) **[15 Marks]** Simplify the utility function. Write U as a function of Q_1 only, find the first-order condition for utility maximization and solve for the optimal Q_1 and Q_2. ii) **[5 Marks]** Verify the second-order condition for a global utility maximum.

Answer: i) **[15 Marks]** We have

$$\begin{aligned} U &= \ln\left(e^{2\ln(Q_2)} \ln\left(e^{Q_1}\right)\right) + 2\ln\left(\sqrt{e^{\ln(Q_2)}}\right) = \ln\left(e^{2\ln(Q_2)}\right) + \ln\left(\ln\left(e^{Q_1}\right)\right) + 2\ln\left((Q_2)^{\frac{1}{2}}\right) \\ &= 2\ln(Q_2) + \ln(Q_1) + \ln(Q_2) = \ln(Q_1) + 3\ln(Q_2). \end{aligned}$$

The budget line is

$$Y = P_1 Q_1 + P_2 Q_2 \Longrightarrow Q_2 = \frac{Y}{P_2} - \frac{P_1}{P_2} Q_1$$

and so

$$U(Q_1) = \ln(Q_1) + 3\ln\left(\frac{Y}{P_2} - \frac{P_1}{P_2} Q_1\right).$$

Thus

$$U'(Q_1) = \frac{1}{Q_1} - \frac{P_1}{P_2} \frac{3}{\frac{Y}{P_2} - \frac{P_1}{P_2} Q_1} = \frac{1}{Q_1} - \frac{3P_1}{Y - P_1 Q_1}.$$

The first-order condition $U'(Q_1^*) = 0$ then yields

$$\begin{aligned} \frac{1}{Q_1^*} - \frac{3P_1}{Y - P_1 Q_1^*} &= 0 \Longrightarrow \frac{1}{Q_1^*} = \frac{3P_1}{Y - P_1 Q_1^*} \Longrightarrow Y - P_1 Q_1^* = 3P_1 Q_1^* \\ &\Longrightarrow 4P_1 Q_1^* = Y \Longrightarrow Q_1^* = \frac{Y}{4P_1}. \end{aligned}$$

As well

$$Q_2^* = \frac{Y}{P_2} - \frac{P_1}{P_2} Q_1^* = \frac{Y}{P_2} - \frac{P_1}{P_2} \frac{Y}{4P_1} = \frac{3}{4} \frac{Y}{P_2}.$$

ii) **[5 Marks]** We have

$$U''(Q_1) = -\underbrace{\frac{1}{Q_1^2}}_{+} - \underbrace{\frac{3P_1^2}{(Y - P_1 Q_1)^2}}_{+} < 0$$

for *all* Q_1 so that $U(Q_1)$ is globally concave and $Q_1^* = \frac{Y}{4P_1}$ yields a global utility maximum.