Only nonprogrammable calculators are allowed.
Duration: 50 minutes. **Total marks: 30**

NAME (in ink): **STUDENT NO (in ink):**

PART I: [6] True/False questions. Circle the correct answer in ink.

1. If $\{v_1, v_2, v_3\}$ is independent, so is $\{v_1 - v_3\}$. [**True** | False]
2. Basis is a linearly independent set of the vector space that is as large as possible. [**True** | False]
3. Three vectors in P_2 can be linearly independent. [**True** | False]

PART II: [6] Multiple choice questions. Circle the correct answer in ink.

1. Let A be a 2×4 matrix. The smallest possible value of nullity A equals to

a) 0 b) 1 c) 4 **d)** none of the above

2. Which of the following is a basis of P_2?

$B_1 = \{x, 1+x, x-x^2\}$, $B_2 = \{1-x, 1-x^2, x+x^2\}$

$B_3 = \{1, 1+x+x^2\}$, $B_4 = \{1, 2-x, 3-x^2, x+2x^2\}$

a) B_1 And B_3 b) B_2 And B_4 c) B_1, B_2 and B_4 **d)** none of the above

3. Let $S = \left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}x\\1\\0\end{bmatrix}, \begin{bmatrix}0\\2\\3\end{bmatrix}\right\}$. Which values of x makes S linearly dependent?

a) $x=0$ b) $x=1$ c) $x=2$ **d)** none of the above

PART III: [18] Long answer questions. Show all your work.

[4] 1. Find a basis of M_{22} containing the linearly independent set $\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}-1&0\\0&1\end{bmatrix}\right\}$.

ANS: The vectors $\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}-1&0\\0&1\end{bmatrix}\right\}$ are linearly independent and $\dim M_{22} = 4$

A vector from the standard basis $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ of M_{22} can be added to extend the given set to form a basis of M_{22}.

Any linear combination of the three given set is given by

$a\begin{bmatrix}1&0\\0&1\end{bmatrix} + b\begin{bmatrix}0&1\\1&0\end{bmatrix} + c\begin{bmatrix}-1&0\\0&1\end{bmatrix} = \begin{bmatrix}a-c&b\\b&a+c\end{bmatrix}$. In order to check which of the vectors from the standard basis is not in the span of the three given vectors (i.e. not a linear combination), we can solve the following four simultaneous equations.

$$\left[\begin{array}{cccc|cccc}1&0&-1&0&1&0&0&0\\0&1&0&0&0&1&0&0\\0&1&0&0&0&0&1&0\\1&0&1&0&0&0&0&1\end{array}\right] \sim \left[\begin{array}{cccc|cccc}1&0&-1&0&1&0&0&0\\0&1&0&0&0&1&0&0\\0&0&2&0&-1&0&0&1\\0&0&0&0&-1&0&0&1\end{array}\right]$$

Any one of the two vectors $\left\{\begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}\right\}$ can be added to get the basis as these two vectors are not in the span of the given vectors.

[7] 2. Let $A=\begin{bmatrix}1&-2&1&1&2\\-1&3&0&2&-2\\0&1&1&3&4\\1&2&5&13&5\end{bmatrix}\sim\begin{bmatrix}1&-2&1&1&2\\0&1&1&3&0\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}=B$.

(a) [2] Find a basis for the column space of A .

ANS: From the row echelon form of A, the columns #1, #2,#5 have pivots. Therefore a basis for the column space of A is

$$\left\{\begin{bmatrix}1\\-1\\0\\1\end{bmatrix},\begin{bmatrix}-2\\3\\1\\2\end{bmatrix},\begin{bmatrix}2\\-2\\4\\5\end{bmatrix}\right\}$$

(b) [3] Find a basis for the null space of A.

ANS: From the row echelon form of A, x_3, x_4 are free variables.

Let $x_3=s, x_4=t$

Also $x_5=0$.

From the second row, we have $x_2+x_3+3x_4=0\Rightarrow x_2=-s-3t$

From the first row, we have $x_1-2x_2+x_3+x_4+2x_5=0\Rightarrow x_1=2(-s-3t)-s-t=-3s-7t$

Therefore the solution of $AX=0$ is given by

$X=\begin{bmatrix}-3s-7t\\-s-3t\\s\\t\\0\end{bmatrix}=-s\begin{bmatrix}3\\1\\-1\\0\\0\end{bmatrix}-t\begin{bmatrix}7\\3\\0\\-1\\0\end{bmatrix}$. So a basis of the null space of A is $\left\{\begin{bmatrix}3\\1\\-1\\0\\0\end{bmatrix},\begin{bmatrix}7\\3\\0\\-1\\0\end{bmatrix}\right\}$

(c) [2] Find the values of $rank\ A$ and $Nullity\ A$.

ANS: $rank\ A=3$ $Nullity\ A=2$

[4] 3. Let $\{v_1,v_2,v_3\}$ be a basis for a vector space V .

a) [3] Show that $S=\{v_1,v_1+v_2,v_1+v_2+v_3\}$ is linearly independent.

ANS:

Let $sv_1+t(v_1+v_2)+u(v_1+v_2+v_3)$ be a linear combination of $v_1,v_1+v_2,v_1+v_2+v_3$

$sv_1+t(v_1+v_2)+u(v_1+v_2+v_3)=(s+t+u)v_1+(t+v)v_2+uv_{31}$

Since $\{v_1,v_2,v_3\}$ is a basis of V, they are linearly independent. Hence $\left.\begin{matrix}s+t+u=0\\t+u=0\\u=0\end{matrix}\right\}\Rightarrow s=t=u=0$

Therefore $S=\{v_1,v_1+v_2,v_1+v_2+v_3\}$ is linearly independent.

b) [1] Is S a basis for V ? Explain.

ANS: yes, since dim(V)=3 and S has 3 linearly independent vectors. Therefore they span V and form a basis.

[3] 4. Determine whether $W=\left\{\begin{bmatrix}a&b\\b&2a\end{bmatrix}\right\}$ is a subspace of M_{22}. Explain.

ANS #1:

Any vector in W can be expressed as $\begin{bmatrix}a&b\\b&2a\end{bmatrix}=a\begin{bmatrix}1&0\\0&2\end{bmatrix}+b\begin{bmatrix}0&1\\1&0\end{bmatrix}$.

i.e. $W=span\left\{\begin{bmatrix}1&0\\0&2\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix}\right\}$, two vectors of M_{22}.

Therefore, W is a subspace of M_{22}.

ANS #2:

Show zero vector is in W.

Show that W is closed under vector addition.

Show that W is closed under scalar multiplication. Conclude that it is a subspace.