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CARLETON UNIVERSITY

FINAL
EXAMINATION
AUGUST 2005

DURATION: 3 HOURS

Department Name and Course Number: Mathematics and Statistics, MATH 3705A
Course Instructor: Dr. J. Abdulrahman

AUTHORIZED MEMORANDA
NONE

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Instructions:

1. Please provide your name, student number and signature below.

Last Name _____ Given Names _____

Signature _____ Student Number _____

2. This examination contains 17 pages. Please report any missing pages to the proctor. Pages 15, 16 and 17 contain tables of Laplace and Fourier transforms and a summary of Fourier series. These pages may be detached for easier viewing.
3. Problems 1 - 14 are multiple choice, worth 3 marks each. Circle the correct answer. Only the answer will be marked. In problems 15 - 20, show your work.

Problem	Maximum Mark	Mark
1 - 14	42	
15	10	
16	10	
17	9	
18	10	
19	9	
20	10	
Total	100	

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1. $\mathcal{L}\{e^{-2t} \cos(3t)\} =$

$$\frac{e^{-2s}}{s^2 + 9}$$

(b) $\frac{s - 2}{(s - 2)^2 + 9}$

(c) $\frac{s + 2}{(s + 2)^2 + 9}$

$$\frac{s + 2}{s^2 + 9}$$

(e) None of the above.

2. $\mathcal{L}\{t \sin(2t)\} =$

(a) $\frac{2}{(s - 1)^2 + 4}$

(b) $\frac{s}{(s^2 + 4)^2}$

(c) $\frac{-s}{(s^2 + 4)^2}$

(d) $\frac{4s}{(s^2 + 4)^2}$

(e) None of the above.

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$$3. \mathcal{L}^{-1} \left\{ \frac{5e^{-4s}}{s^2 - s - 6} \right\} =$$

- (a) $u(t-4) [e^{3t} - e^{-2t}]$
- (b) $u(t-3)e^{3t} - u(t-2)e^{-2t}$
- (c) $e^{3(t-4)} - e^{-2(t-4)}$
- (d) $u(t-4) [e^{3(t-4)} - e^{-2(t-4)}]$
- (e) None of the above.

$$4. \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 2s + 5} \right\} =$$

- (a) $e^{-t} \sin(2t)$
- (b) $e^{-t} [\cos(2t) - \frac{1}{2} \sin(2t)]$
- (c) $e^{-t} \cos(2t) - e^{-t} \sin(2t)$
- (d) $e^{-t} \cos(2t)$
- (e) None of the above.

5. The general solution of the differential equation $4x^2y'' + y = 0$, valid for $x \neq 0$, is given by

(a) $c_1 \cos\left(\frac{1}{2} \ln|x|\right) + c_2 \sin\left(\frac{1}{2} \ln|x|\right)$

(b) $c_1 \cos\left(\frac{1}{2}x\right) + c_2 \sin\left(\frac{1}{2}x\right)$

(c) $c_1|x|^{\frac{1}{2}} + c_2|x|^{\frac{1}{2}}$

(d) $|x|^{\frac{1}{2}}(c_1 + c_2 \ln|x|)$

(e) None of the above.

6. The general solution of the differential equation $x^2y'' + 2xy' + y = 0$, valid for $x \neq 0$, is given by

(a) $c_1|x|^{-1} + c_2|x|^{-1}$

(b) $|x|^{-1}(c_1 + c_2 \ln|x|)$

(c) $|x|^{-\frac{1}{2}} \left[c_1 \cos\left(\frac{\sqrt{3}}{2} \ln|x|\right) + c_2 \sin\left(\frac{\sqrt{3}}{2} \ln|x|\right) \right]$

(d) $|x|^{-\frac{1}{2}} \left[c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right) \right]$

(e) None of the above.

7. The general solution of the differential equation $x^2y'' + xy' + (5x^2 - 9)y = 0$, valid for $x > 0$, is given by

- (a) $c_1J_{\sqrt{5}}(3x) + c_2J_{-\sqrt{5}}(3x)$
- (b) $c_1J_3(\sqrt{5}x) + c_2Y_3(\sqrt{5}x)$
- (c) $c_1J_{\sqrt{5}}(3x) + c_2Y_{\sqrt{5}}(3x)$
- (d) $c_1J_3(\sqrt{5}x) + c_2J_{-3}(\sqrt{5}x)$
- (e) None of the above.

8. Let $f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 2, & 1 \leq x \leq 2 \end{cases}$ on $[0, 2]$. At $x = 79$, the Fourier sine series of f converges to

- (a) 2
- (b) $\frac{3}{2}$
- (c) 0
- (d) $-\frac{3}{2}$
- (e) None of the above.

9. The differential equation $x^2y'' + 3xy' + y + \lambda y = 0$, when placed in the Sturm-Liouville form $(py')' - qy + \lambda ry = 0$, has the weight function $r(x) =$

- (a) $xe^{\frac{3}{2}x^2}$
- (b) $e^{\frac{3}{2}x^2}$
- (c) x^2
- (d) x
- (e) None of the above.

10. Given the Bessel identity $\frac{1}{\alpha} \frac{d}{dx} [x^\nu J_\nu(\alpha x)] = x^\nu J_{\nu-1}(\alpha x)$, $\nu > 0$, $\alpha \neq 0$, $\int_0^2 x^3 J_0(2x) dx =$

- (a) $8J_1(4) - 4J_0(4)$
- (b) $4J_1(4) - 2J_2(4)$
- (c) $8J_0(4)$
- (d) $8J_1(4)$
- (e) None of the above.

11. $\mathcal{F}\{e^{2ix-|x+3|}\} =$

(a) $\frac{2e^{-3i(\lambda+2)}}{1+\lambda^2}$

(b) $\frac{2e^{-3i(\lambda+2)}}{1+(\lambda+2)^2}$

(c) $\frac{2e^{-3i\lambda}}{1+(\lambda+2)^2}$

(d) $\frac{2e^{-3i\lambda}}{1+(\lambda-2)^2}$

(e) None of the above.

12. $\mathcal{F}\{xe^{-2x^2}\} =$

(a) $\frac{\sqrt{\pi}}{2}e^{-2\lambda^2}$

(b) $\frac{1}{2\pi}\lambda e^{-2\lambda^2}$

(c) $\frac{\sqrt{\pi}}{2}\lambda e^{-\frac{\lambda^2}{8}}$

(d) $\frac{i\sqrt{\pi}}{4\sqrt{2}}\lambda e^{-\frac{\lambda^2}{8}}$

(e) None of the above.

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$$13. \mathcal{F}^{-1} \left\{ \frac{e^{3i\lambda}}{1 + (\lambda + 2)^2} \right\} =$$

$$(a) \frac{1}{2} e^{2i(x-3) - |x-3|}$$

$$(b) \frac{1}{2} e^{2ix - |x-3|}$$

$$(c) \frac{1}{2} e^{2i(x-3) - |x|}$$

$$(d) \frac{1}{2} e^{2i(x+3) - |x+3|}$$

(e) None of the above.

$$14. \mathcal{F}^{-1} \{ \lambda e^{-|\lambda|} \} =$$

$$(a) \frac{x}{\pi(1+x^2)}$$

$$\frac{-x}{\pi(1+x^2)}$$

$$\frac{2ix}{\pi(1+x^2)^2}$$

$$(d) \frac{-2ix}{\pi(1+x^2)^2}$$

(e) None of the above.

15. Employ the Laplace transform to solve the initial-value problem

$$y'' + y = u(t - 2) \sin(t - 2), \quad y(0) = 1, \quad y'(0) = -1, \quad \text{where } u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

16. Find one (non-zero) series solution y_1 of the differential equation $2xy'' + y' - y = 0$, valid near $x_0 = 0$.

17. Suppose that f is odd and 2-periodic, with $f(x) = x^2$ on $[0, 1]$. Find the Fourier series of f .

18. The solution of the heat equation $u_{xx} = \frac{1}{\alpha^2}u_t$, $0 < x < L$, which satisfies the boundary conditions $u(0, t) = 0$, $u(L, t) = 0$, has the form

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t}$$

Find the solution of $u_{xx} = u_t$, $0 < x < \pi$, which satisfies the boundary conditions $u(0, t) = 0$, $u(\pi, t) = \pi$, and the initial condition $u(x, 0) = x + 1$. Write down the complete solution $u(x, t)$.

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19. The bounded solution of Laplace's equation $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ outside the circle $r = a$ has the form

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n} [a_n \cos(n\theta) + b_n \sin(n\theta)].$$

Find the bounded solution of Laplace's equation outside the circle $r = 2$, which satisfies the boundary condition $u(2, \theta) = 3 - 2 \cos(3\theta) + 4 \sin(2\theta)$. Write down the complete solution $u(r, \theta)$.

20. Find all eigenvalues and corresponding eigenfunctions of the Sturm-Liouville problem
 $y'' + 2y' + \lambda y = 0$, $y(0) + y'(0) = 0$, $y(1) = 0$.

Table of Laplace Transforms

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st} dt, \quad s > 0$$

$$\mathcal{L}\{t^p\} = \frac{\Gamma(p+1)}{s^{p+1}}, \quad p > -1, \quad \text{and } \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \text{ if } n \geq 0 \text{ is an integer}$$

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2 + a^2}$$

$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2 + a^2}$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a), \quad s > a$$

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s), \quad s > a \geq 0$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0), \quad n \geq 0$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s) \equiv (-1)^n \frac{d^n}{ds^n} F(s), \quad n \geq 0$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(x) dx$$

$$\mathcal{L}\left\{\int_0^t f(x) dx\right\} = \frac{1}{s} F(s)$$

$$\mathcal{L}\{f(t) * g(t)\} \equiv \mathcal{L}\left\{\int_0^t f(t-x)g(x) dx\right\} = F(s)G(s), \quad \text{where } G(s) = \mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}, \quad a \geq 0$$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-\omega s}} \int_0^{\omega} e^{-st} f(t) dt \quad \text{whenever } f \text{ is periodic with period } \omega$$

Summary of Fourier Series

1. The Fourier sine series of a function f defined on $[0, L]$ is given by

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1.$$

2. The Fourier cosine series of a function f defined on $[0, L]$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 0.$$

3. The full Fourier series of a function f defined on $[-L, L]$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right],$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 0, \quad b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1.$$

4. The full Fourier series of a function f defined on $[0, 2L]$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right],$$

$$a_n = \frac{1}{L} \int_0^{2L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 0, \quad b_n = \frac{1}{L} \int_0^{2L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1.$$

5. The Fourier series of an ω -periodic function f on $\mathbb{R} = (-\infty, \infty)$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2n\pi x}{\omega}\right) + b_n \sin\left(\frac{2n\pi x}{\omega}\right) \right],$$

$$\begin{aligned} a_n &= \frac{1}{L} \int_0^{2L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{\omega} \int_0^{\omega} f(x) \cos\left(\frac{2n\pi x}{\omega}\right) dx \\ &= \frac{2}{\omega} \int_{\alpha}^{\alpha+\omega} f(x) \cos\left(\frac{2n\pi x}{\omega}\right) dx, \quad n \geq 0, \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{L} \int_0^{2L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{\omega} \int_0^{\omega} f(x) \sin\left(\frac{2n\pi x}{\omega}\right) dx \\ &= \frac{2}{\omega} \int_{\alpha}^{\alpha+\omega} f(x) \sin\left(\frac{2n\pi x}{\omega}\right) dx, \quad n \geq 1, \end{aligned}$$

where $L = \frac{\omega}{2}$ and α is any real number.

Table of Fourier Transforms

$$\mathcal{F}\{f(x)\} = \hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x) e^{i\lambda x} dx$$

$$\mathcal{F}^{-1}\{F(\lambda)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda) e^{-i\lambda x} d\lambda$$

$$\mathcal{F}\{u(x-a) - u(x-b)\} = \frac{1}{i\lambda} [e^{i\lambda b} - e^{i\lambda a}], \quad a < b$$

$$\mathcal{F}\{e^{-|x|}\} = \frac{2}{1 + \lambda^2}$$

$$\mathcal{F}\{\delta(x-a)\} = e^{i\lambda a}$$

$$\mathcal{F}\{e^{iax} f(x)\} = \hat{f}(\lambda + a)$$

$$\mathcal{F}\{f(x-a)\} = e^{i\lambda a} \hat{f}(\lambda)$$

$$\mathcal{F}\{f'(x)\} = -i\lambda \hat{f}(\lambda)$$

$$\mathcal{F}\{xf(x)\} = -i \frac{d\hat{f}}{d\lambda}$$

$$\mathcal{F}\{f(\alpha x)\} = \frac{1}{|\alpha|} \hat{f}\left(\frac{\lambda}{\alpha}\right), \quad \alpha \neq 0$$

$$\mathcal{F}\{e^{-tx^2}\} = \sqrt{\frac{\pi}{t}} e^{-\frac{\lambda^2}{4t}}, \quad t > 0$$

$$\mathcal{F}\{(f * g)(x)\} \equiv \mathcal{F}\left\{\int_{-\infty}^{\infty} f(s)g(x-s)ds\right\} = \hat{f}(\lambda)\hat{g}(\lambda), \quad \text{where } \hat{g}(\lambda) = \mathcal{F}\{g(x)\}$$