1

section 1.1

1. $(1-x)y'' - 4xy' + 5y = \cos(x)$

$(1-x)\frac{d^2y}{dx^2} - 4x + 5y = \cos(x)$

$(1-x)\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 5y - \cos(x) = 0$

linear second order

2. $x\frac{dy^3}{dx^3} = \left(\frac{dy}{dx}\right)^4 + y = 0$

linear ; third order

3. $t^5 y^{(4)} - t^3 y'' + 6y = 0$

linear ; fourth order

5. $\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

non linear ; second order

6. $\frac{d^2R}{dt^2} + kxR^2 = 0$ non linear ; second order

7. $(\sin\theta)y''' - (\cos\theta)y' = 2$

$\Longleftrightarrow (\sin\theta)\frac{d^3y}{dx} - 2\cos(\theta)\frac{dy}{dx} = 0$

linear ; third order.

8. $\ddot{x} - (1 - \frac{1}{3}\dot{x}^2)\dot{x} + x = 0$

$\frac{d^2y}{dx^2} - (1 - 1/3(\frac{dy}{dx})^2)\frac{dy}{dx} + x = 0$ non linear ; second order.

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10. $dv + (v + uv - ue^u)\,du = 0$

non linear in u
linear in v.

11. $2y' + y = 0$ $\quad y = e^{-x/2}$

$2(e^{-x/2})' + e^{-x/2} = 0$

$-e^{-x/2} + e^{-x/2} = 0$

13. $y'' - 6y' + 13y = 0$ $\quad y = e^{3x}\cos 2x$

$y' = 3e^{x} \times \cos 2x + e^{3x}(-2\sin 2x)$

$y'' - 6y' + 13y = 0$

$\longrightarrow$ next page.

$\cancel{y' = 3e^{3x} \times \cos 2x + e^{3x}(-2\sin 2x)}$
$\cancel{= u(x) + v(x)}$

$\cancel{y'' = 9e^{3x} \times \cos 2x + 3e^{3x}(-2\sin 2x) + 3e^{3x}(-2\sin 2x)}$
$\cancel{+ e^{3x}(-4\cos 2x)}$

③

13. $y'' - 6y' + 13y = 0$

$y'' - 6\left(3e^{3x} \times \cos 2x + e^{3x}(-2\sin 2x)\right)$
$+ 13(e^{3x}\cos 2x) = 0$

$y' = 3e^{3x} \times \cos 2x + e^{3x}(-2\sin 2x)$
$= u(x) + v(x)$

$\rightarrow u(x) = 3e^{3x} \times \cos 2x$
$= a(x) \times b(x)$

$v(x) = e^{3x}(-2\sin 2x)$

- $a(x) = 3e^{3x}$
  $a'(x) = 9e^{3x}$
- $b(x) = \cos 2x$
  $b'(x) = -2\sin(2x)$

$u'(x) = a'b + ab'$
$= 9e^{3x} \times \cos 2x + 3e^{3x} \times (-2\sin 2x)$

$\rightarrow v(x) = e^{3x}(-2\sin 2x)$
$= a \times b$

$a = e^{3x}$ $a' = 3e^{3x}$
$b = -2\sin 2x$ $b' = -4\cos 2x$

$v'(x) = a'b + ab'$
$= 3e^{3x}(-2\sin 2x) + e^{3x}(-4\cos 2x)$

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$$y'' - 6y' + 13y = 0$$

$\Leftrightarrow$ $9e^{3x} \times \cos 2x + 3e^{3x}(-2\sin 2x) + 3e^{3x}(-2\sin 2x)$
$+ e^{3x}(-4\cos 2x)$
$- 6(3e^{3x} \times \cos 2x) + e^{3x}(-2\sin 2x)$
$+ 13(e^{3x} \cos 2x) = 0$

$\Leftrightarrow$ $e^{3x}(9\cos 2x - 6\sin 2x - 6\sin 2x - 4\cos 2x)$
$-(18e^{3x}\cos 2x - 12e^{3x}\sin 2x) + 13e^{3x}\cos 2x = 0$

$\Leftrightarrow$ $e^{3x}(5\cos 2x - 12\sin 2x) - 18e^{3x}\cos 2x + 12e^{x}\sin 2x$
$+ 13e^{3x}\cos 2x = 0$

$\Leftrightarrow$ $5e^{3x}\cos 2x - 12\sin 2x e^{3x} + 12\sin 2x e^{3x} - 18e^{x}\cos 2x + 13e^{3x}\cos 2x$
$= 0$

14: $y'' + y = \tan(x)$ $y = -(\cos x)$
$\ln(\sec x + \tan x)$

$= u(x) \times v(x)$
$= u'v + uv'$

- $u(x) = -\cos(x)$
$u'(x) = \sin(x)$

- $v(x) = \ln(\sec x + \tan x)$
$= \ln(a(x))$ $a(x) = \sec x + \tan x$
$v'(x) = \frac{a'(x)}{a(x)}$ $a'(x) = \sec x \times \tan x$
$+ \sec^2 x$.

$v'(x) = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \frac{\sec(x)(\tan x + \sec x)}{\sec(x) + \tan(x)}$

$\boxed{v'(x) = \sec x}$

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$\rightarrow$ 14.

$$y' = \sin(x) \times \ln(\sec(x + \tan x) - (\cos(x)) \times \sec(x)$$

$$y' = u(x) + v(x)$$

$u(x) = \sin(x) \times \ln(\sec(x \tan x)$

$a(x) = \sin(x)$ $\quad$ $b(x) = \ln(\sec \tan)$

$a'(x) = \cos(x)$ $\quad$ $b'(x) = \sec x.$

$u'(x) = a(x) \times b(x)$

$= a'b + ab'$

$= \cos(x) \ln(\sec(x) \tan(x)) + \sin(x) \times \sec(x)$

$v(x) = -\cos x \times \sec(x)$

$= c(x) \times d(x)$.

$c(x) = -\cos(x)$ $\quad$ $d(x) = \sec(x)$

$c'(x) = +\sin(x)$ $\quad$ $d'(x) = \sec x \tan$

$= c'd + cd'$

$= +\sin(x)\sec(x) - \cos(x)\sec(x) \times \tan(x)$

$y'' = u'(x) + v'(x)$

$= \cos(x) \ln(\sec(x) \tan(x)) + \sin(x) \times \sec(x) + \sin(x)\sec(x) - \cos(x)\sec(x) \times \tan(x)$

$$y'' + y = \tan(x)$$

$\Leftrightarrow$ $\cos(x)\ln(\sec(x)\tan(x)) + \sin(x)\sec(x) + \sin(x)\sec(x) - \cos(x)\sec(x)\tan(x)$

$-\cos(x)\ln(\sec(x)\tan(x)) = \tan(x)$

$\Leftrightarrow$ $2(\sin(x)\sec(x)) - \cos(x)\sec(x)\tan(x) = \tan(x)$

$\Leftrightarrow$ $2\left(\sin(x) \times \frac{1}{\cos(x)}\right) - \left(\cos(x) \times \frac{1}{\cos(x)}\right) \times \tan(x) = \tan(x)$

$\Leftrightarrow$ $2\tan(x) - \tan(x) = \tan(x)$ $\longrightarrow$ $\tan(x) = \tan(x)$.

⑥

21.

$$\frac{dP}{dt} = P(1-P) \qquad P = \frac{c_1e^t}{1+c_1e^t}$$

$$P = \frac{u(t)}{v(t)} \qquad u(t) = c_1e^t$$
$$u'(t) = c_1e^t$$

$$v(t) = 1 + c_1e^t$$
$$v'(t) = c_1e^t$$

$$\frac{dP}{dt} = \frac{u'(t)v(t) - u(t)v'(t)}{v(t)^2}$$

$$= \frac{(c_1e^t)(1+c_1e^t) - (c_1e^t)(c_1e^t)}{(1+c_1e^t)^2}$$

$$= \frac{c_1e^t + (c_1e^t)^2 - (c_1e^t)^2}{(1+c_1e^t)^2}$$

$$\boxed{\frac{dP}{dt} = \frac{c_1e^t}{(1+c_1e^t)^2}}$$

$$P(1-P) = \frac{c_1e^t}{1+c_1e^t}\left(1 - \frac{c_1e^t}{1+c_1e^t}\right)$$

$$= \frac{c_1e^t}{1+c_1e^t} - \left(\frac{c_1e^t}{1+c_1e^t}\right)^2$$

$$= \frac{c_1e^t(1+c_1e^t)}{(1+c_1e^t)^2} - \frac{(c_1e^{t^2})}{(1+c_1e^t)^2}$$

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$$= \frac{c_1 e^t + (c_1 e^t)^2}{(1+c_1e^t)^2} - \frac{(c_1e^t)^2}{(1+c_1e^t)^2}$$

$$= \frac{c_1 e^t}{(1+c_1e^t)^2}$$

$$\boxed{P(1-P) = \frac{c_1e^t}{(1+c_1e^t)^2}}$$

$$\frac{dP}{dt} = \frac{c_1e^t}{(1+c_1e^t)^2}$$

$$P(1-P) = \frac{c_1e^t}{(1+c_1e^t)^2}$$

thus: $\frac{dP}{dt} = P(1-P)$.

23:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0 \quad ; \quad y = c_1e^{2x} + c_2xe^{2x}$$

$$y' = 2c_1e^{2x} + (c_2e^{2x} + 2c_2xe^{2x})$$
$$= 2c_1e^{2x} + c_2e^{2x}(1+2x)$$

$$y'' = 4c_1e^{2x} + 2c_2e^{2x} + (1+2x)2c_2e^{2x}$$
$$= 4c_1e^{2x} + c_2e^{2x}(2+2+4x)$$

$y'' - 4y' + 4y$

$$\Leftrightarrow 4c_1e^{2x} + 2c_2e^{2x} + 2c_2e^{2x} + 4xc_2e^{2x} - 8c_1e^{2x} - 4c_2e^{2x} - 8xc_2e^{2x} + 4c_1e^{2x} + 4c_2xe^{2x} = 0$$

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$y'' - 4y' - 4y = 0$

y is a solution of the given D.E

$y \in I$; $I = (-\infty, +\infty)$

24.

$y = c_1 x^{-1} + c_2 x + c_3 x \ln(x) + 4x^2$

$y' = -c_1 x^{-2} + c_2 + c_3 \ln(x) + \frac{1}{x} c_3 x + 8x$

$= -c_1 x^2 + c_2 + c_3 \ln(x) + c_3 + 8x$

$= -c_1 x^2 + c_2 + c_3 \ln(x-1) + 8x$

$y'' = 2c_1 x^{-3} + c_3 x^{-1} + 8$

$\boxed{y^{(3)} = -6 c_1 x^{-4} - c_3 x^{-2}}$

$x^3 y^{(3)} + 2x^2 y'' - x y' + y$

$= x^3(-6c_1 x^{-4} - c_3 x^{-2}) - x(-c_1 x^{-2} + c_2 + c_3 \ln(x) + c_3 + 8x)$
$+ c_1 x^{-1} + c_2 x + c_3 x \ln(x) + 4x^2$.

$= -6c_1 x^{-1} - c_3 x + 4x^{-1} c_1 + 2c_3 x + 16x^2$
$- c_2 x + c_1 x^{-1} - x c_3 \ln(x) - c_3 x - 8x^2$
$+ c_1 x^{-1} + c_2 x + c_3 x \ln(x) + 4x^2$

$= +12x^2$

y is a solution; $I = (-\infty ; +\infty)$

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$x = c_1 \cos t + c_2 \sin t$ $\quad x'' + x = 0$

$x(0) = -1 \quad x'(0) = 8$

$x' = -c_1 \sin t + c_2 \cos t$ and $-c_1 \sin 0 + c_2 \cos 0 = 8$

$c_1 \cos 0 + c_2 \sin 0 = -1$ $\quad \Leftrightarrow \boxed{c_1 = -8}$

$\Leftrightarrow \boxed{c_2 = -1}$

$$\boxed{x = -8 \cos t - \sin t}$$

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$x = c_1 \cos t + c_2 \sin t$ $\quad x' = -c_1 \sin t + c_2 \cos t$

$x\left(\frac{\pi}{6}\right) = \frac{1}{2} \quad x'\left(\frac{\pi}{6}\right) = 0$

$\frac{1}{2} = c_1 \cos \frac{\pi}{6} + c_2 \sin \frac{\pi}{6}$ and $-c_1 \sin \frac{\pi}{6} + c_2 \cos \frac{\pi}{6} = 0$

$\frac{1}{2} = \frac{\sqrt{3}}{2} c_1 + \frac{1}{2} c_2$ and $-\frac{1}{2} c_1 + \frac{\sqrt{3}}{2} c_2 = 0$

$\Leftrightarrow 1 - \sqrt{3} c_1 = c_2$ and $c_1 = -\sqrt{3} c_2$

$\Leftrightarrow 1 - \sqrt{3}(-\sqrt{3} c_2) = c_2$ $\quad \Leftrightarrow \boxed{c_1 = \frac{\sqrt{3}}{2}}$

$\Leftrightarrow 1 + 3c_2 = c_2$

$\boxed{c_2 = -1/2}$

$$\boxed{x = \frac{\sqrt{3}}{2} \cos \frac{\pi}{6} - \frac{1}{2} \sin t}$$

10

11

$y = c_1 e^{x} + c_2 e^{-x}$

$y'' - y = 0$ and $y(0) = 1$, $y'(0) = 2$

$y(0) = 1 \iff c_1 e^0 + c_2 e^0 = 1$

$\iff c_1 + c_2 = 1$

$y' = c_1 e^{x} - c_2 e^{-x}$

$y'(0) = 2 \iff 2 = c_1 e^0 - c_2 e^0$

$2 = c_1 - c_2$

$\begin{cases} c_1 - c_2 = 2 \\ c_1 + c_2 = 1 \end{cases} \iff \begin{cases} c_1 = 2 + c_2 \\ 2 + c_2 + c_2 = 1 \end{cases} \begin{cases} 2 + 2c_2 = 1 \\ 2(1 + c_2) = 1 \end{cases}$

$\iff \begin{cases} 1 + c_2 = 1/2 \\ c_2 = -1/2 \end{cases}$ and $c_1 - c_2 = 2$

$\iff c_1 - (-1/2) = 2$

$\iff c_1 = \frac{3}{2}$

$\boxed{y = \frac{3}{2} e^{x} - \frac{1}{2} e^{-x}}$

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$$y = c_1 e^x + c_2 e^{-x}$$

$y'' - y = 0$ and $y(1) = 0$ $y'(1) = e$

$y(1) = 0 \iff c_1 e + c_2 e^{-1} = 0$

$y'(1) = e \iff c_1 e - c_2 e^{-1} = e$

$$\begin{cases} c_1 e + c_2 e^{-1} = 0 \; (1) \\ c_1 e - c_2 e^{-1} = e \; (2) \end{cases} \overset{(1+2)}{\iff} \begin{cases} 2c_1 e = e \\ 2c_1 = 1 \iff \boxed{c_1 = 1/2} \end{cases}$$

and $\frac{e}{2} + c_2 e^{-1} = 0 \iff c_2 e^{-1} = \frac{-e}{2}$

$$\iff c_2 = \frac{-e}{2e^{-1}} = \frac{-2e}{2}$$

$$\boxed{= -e}$$

$$y = \frac{e^x}{2} - e \times e^{-x}$$

$$\boxed{y = \frac{1}{2} \times e^x - e^{-x+1}}$$

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17

$$\frac{dy}{dx} = y^{2/3}$$

$$y' = y^{2/3}$$

$$f(x,y) = y^{2/3}$$

$$= \sqrt[3]{y^2}$$

$$\boxed{y > 0 \text{ or } y < 0}$$

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$y' = \sqrt{xy}$

$\frac{\delta f}{\delta y} = \sqrt{x} \cdot \frac{1}{2\sqrt{y}} \sqrt{\frac{x}{y}}$

function has a unique solution when:

$y \neq 0$ and $\begin{cases} x > 0 & ; y > 0 \\ \text{or } x < 0 & ; y < 0 \end{cases}$

section 1.3

10

$A(0) = 50 lb$

$R_{in} = \frac{3}{2} R_{out}$ with

- $R_{in} = 3 \times 2 = 6$ lb / min.
- $R_{out} =$ concentration in out :

$c(t) = \frac{A(r)}{300+r}$ lb / gal.

$R_{out} = \frac{2A}{300+r}$ lb / min

$\frac{dA}{dr} = 6 - \frac{2A}{300r}$

13)

$nA = 2\,in$ water leaving the tank per second $= cA_h\sqrt{2gh}$

$g = 32\,ft/s^2$

$0 < c < 1$

the variation of water in the tank is defined by

$$\frac{dV}{dr} = -c\,A_h\sqrt{2gh}$$

The volume of water at the time (in the tank):

$V(t) = A_w h$

$$\frac{dv}{dt} = A_w\,\frac{dh}{dt}$$

$$\Rightarrow A_w\,\frac{dh}{dr} = -cA_h\sqrt{2gh}$$

for nA, $2in = 0{,}17\,fr$

$A_h = 2\times 0{,}17\pi$

$A_h = 0{,}34\pi\,fr$ and $A_w = 10^2\,fr$

$A_w = 100\,fr$

$$\frac{dh}{dr} = -c\,0{,}0034\pi\sqrt{64h}$$

$$\frac{dh}{dr} = -c\,0{,}0272\pi\sqrt{h}$$