ENGR 233

Section 7.1 – Vectors in 2-Space

Two vectors are parallel iff they are non-zero scalar multiples of each other.

Vector $\overrightarrow{AB}$ is usually found by using the points $B - A$

Let $a =< a_1, a_2 >$ and $b =< b_1, b_2 >$ be vectors.

- Addition

$$a + b = < a_1 + b_1, a_2 + b_2 >$$

- Subtraction

$$a - b = < a_1 - b_1, a_2 - b_2 >$$

- Scalar Multiplication

$$ka = < ka_1, ka_2 >$$

- Equality

$$a = b \text{ iff } a_1 = b_1 \ \& \ a_2 = b_2$$

Vector Properties (really obvious ones excluded):

$$a + b = b + a$$

$$a + (b + c) = (a + b) + c$$

$$k(a + b) = ka + kb$$

$$(k_1 + k_2)a = ak_1 + ak_2$$

Magnitude:

$$\|a\| = \sqrt{{a_1}^2 + {a_2}^2}$$

Unit Vector (Normalization): A vector with a magnitude of one unit.

$$\hat{u} = \frac{a}{\|a\|}$$

If a and b are vectors and c_1 and c_2 are scalars then the linear combination is written as follows:

$$c_1 a + c_2 b$$

Section 7.2 – Vectors in 3-Space

Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$

3D-Distance Formula:

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Midpoint Formula:

$$M(P_1, P_2) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Position Vector: The vector which begins at the origin and ends at the desired point

2-Space vector properties and unit vector methods apply to 3-Space vectors.

Section 7.3 – Dot Product

Dot Product: Denoted by $a \cdot b$ and results in a scalar

2-Space:

$$a \cdot b = a_1 b_1 + a_2 b_2$$

3-Space:

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Using Angles:

$$a \cdot b = \|a\| \|b\| \cos\theta$$

Dot Product Properties:

$$a \cdot b = 0 \text{ if } a = 0 \text{ or } b = 0$$

$$a \cdot b = b \cdot a$$

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

$$a \cdot (kb) = k(a \cdot b)$$

$$a \cdot a \geq 0$$

$$a \cdot a = \|a\|^2$$

Orthogonal Vectors: If a and b are non-zero vectors:

$a \cdot b > 0$ iff θ is acute

$a \cdot b < 0$ iff θ is obtuse

$a \cdot b = 0$ iff $\cos \theta = 0$

If the dot product is zero ($a \cdot b = 0$), vectors are *perpendicular*.

Angle between two vectors:

$$\cos(\theta) = \frac{a \cdot b}{\|a\|\|b\|} = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\|a\|\|b\|}$$

Direction Cosines: For a nonzero vector $a = a_1 i + a_2 j + a_2 k$ in 3-Space, the angles α, β and γ between a and the unit vectors i, j, and k respectively (the axes) are called the direction angles of a. From trig:

$$\cos(\alpha) = \frac{a_1}{\|a\|}$$

$$\cos(\beta) = \frac{a_2}{\|a\|}$$

$$\cos(\gamma) = \frac{a_3}{\|a\|}$$

The direction cosines of a vector are simply the components of the unit vector. Because of this:

$$\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = 1$$

Component a on b: Returns a scalar

$$comp_b a = \|a\| \cos(\theta) = \frac{\|a\|\|b\| \cos(\theta)}{\|b\|} = \frac{a \cdot b}{\|b\|} = a \cdot \left(\frac{b}{\|b\|}\right) = \frac{a \cdot b}{\|b\|} = a \cdot u_b$$

Projection of a onto b: Returns a vector

$$proj_a b = (comp_b b)\left(\frac{b}{\|b\|}\right) = \left(\frac{a \cdot b}{b \cdot b}\right) b$$

Work:

$$W = F \cdot d$$

Section 7.4 – Cross Product

Calculating Determinants:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Cross Product:

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Cross Product Properties:

$$a \times b = 0 \text{ if } a = 0 \text{ or } b = 0$$

$$a \times b = -b \times a$$

$$a \times (b + c) = (a \times b) + (a \times c)$$

$$a \times a = 0$$

$$a \cdot (a \times b) = 0$$

$$b \cdot (a \times b) = 0$$

$$a \times b \neq b \times a$$

The cross product of two vectors gives another vector that is perpendicular to the plane created by the original two vectors (right hand rule/screwdriver rule).

Two nonzero vectors are *parallel* iff $a \times b = 0$

Magnitude of the Cross Product:

$$\|a \times b\| = \|a\|\|b\| \sin(\theta) = \text{area of a parallelogram}$$

Scalar Triple Product:

$$a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \text{volume of a parallelipided}$$

Coplanar Vectors: Vectors that lie in the same plane. If $a \cdot (b \times c) = 0$ then the vectors are *coplanar*.

Section 7.5 – Lines and Planes in 3-Space

Vector Equation: Need point it passes through and parallel vector:

$$r = r_2 - ta$$

Where a is a direction vector parallel to the line that goes through the point, r_2 is the position vector of the point the line should pass through, and t is a variable parameter.

Or you can look at it like this: r_2 is the point and a is the direction vector.

https://www.youtube.com/watch?v=H7wre3njI0Y

(Sorry if you printed this out and ^that's useless lol)

Ex: For some vector $V =< x, y, z >$ parallel to vector$a = < a_1, a_2, a_3 >$ that passes through point $P(x_0, y_0, z_0)$

$$V = P + ta$$

$$< x, y, z > = < x_0, y_0, z_0 > \; + \; t < a_1, a_2, a_3 >$$

Parametric Equations: Following from the above example, if you add the two vectors and isolate each component each as a function of t then you get the parametric equations:

$$< x(t), y(t), z(t) > = \; < x_0 + ta_1, \;\; y_0 + ta_2, \;\;\; z_0 + ta_2 >$$

$$x(t) = x_0 + ta_1$$

$$y(t) = y_0 + ta_2$$

$$z(t) = z_0 + ta_2$$

Symmetric Equations: Take the parametric equations and make them all equal to t

$$t = \frac{x - x_0}{a_1} = \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3}$$

Equation of a Plane:

The graph of any equation $ax = by + cz + d = 0$, a, b, c not all zero, is a plane with the normal vector$n = < a, b, c >$.

A plane can be written as a single equation, $a(x - x_0) + \; b(y - y_0) + \; c(z - z_0) = 0$ where (x_0, y_0, z_0) is a point on the plane and vector $n = < a, b, c >$is perpendicular to the plane.

A normal vector to $x + y - 4z = 1$ is $< 1, 1, -4 >$

Intersection Point of Two Lines:

```
          Line1                           Line2
          -----                           -----
      x = x1 + a1 * t1                x = x2 + a2 * t2
      y = y1 + b1 * t1                y = y2 + b2 * t2
      z = z1 + c1 * t1                z = z2 + c2 * t2
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If we set the two x values equal, and the two y values equal we get these two equations.

$$x1 + a1 * t1 = x2 + a2 * t2$$
$$y1 + b1 * t1 = y2 + b2 * t2$$

You can solve these equations for t1 and t2. Then put those values back into the parametric equations to solve for the intersection point.

Section 9.1 – Vector Functions

Curve: A curve C in the xy-plane is just a set of ordered pairs (x, y)

Parametric Curve: If the x and y coordinates of a point on the curve are defined by a pair of functions, $x = f(t)$, $y = g(t)$ that are continuous over some interval $a \leq t \leq b$.

A Parametric curve in space of a Space Curve is a set of ordered triplets (x, y, z) where:

$$x = f(t), y = g(t), z = h(t)$$

Are continuous on an interval defined by $a \leq t \leq b$

Vector Function: Vector whose components are functions of a parameter t

$$r(t) = < f(t), g(t), h(t) >$$

The arrowhead of this vector function $r(t)$ traces out the curve in space.

Differentiation: The derivative of a vector function r is

$$r'(t) = < f'(t), g'(t), h'(t) >$$

Chain Rule: If r is a differentiable function and $s = u(t)$ is a differentiable scalar function, then the derivative of $r(s)$ with respect to t is:

$$\frac{dr}{dt} = \frac{dr}{ds}\frac{ds}{dt} = r'(s)u'(t)$$

Basically it's the tree diagram thing where is one function is made of another function that depends on another variable, you follow it through the branches and derive it as such. Or just replace everything and then use regular old chain rule.

Rules of Differentiation: Let $r_1$and r_2 be differentiable vector functions and $u(t)$ a scalar function:

$$\frac{d}{dt}[r_1(t)+r_2(t)]=r_1'(t)+r_2{}'(t)$$

$$\frac{d}{dt}[u(t)r_1'(t)]=u(t)r_1'(t)+u'(t)r_1(t)$$

$$\frac{d}{dt}[r_1(t)\cdot r_2(t)]=r_1(t)\cdot r_2'(t)+r_1{}'(t)\cdot r_2(t)$$

$$\frac{d}{dt}[r_1(t)\times r_2(t)]=r_1(t)\times r_2'(t)+r_1{}'(t)\times r_2(t)$$

Integration:

$$\int_a^b r(t)\,dt=<\int_a^b f(t)\,dt\,,\int_a^b g(t)\,dt\,,\int_a^b h(t)\,dt>$$

Length of a Space Curve from a to b:

$$s=\int_a^b \sqrt{[f'(t)]^2+[g'(t)]^2+[h'(t)]^2}\;dt=\int_a^b \|r'(t)\|\,dt$$

Section 9.2 – Motion of a Curve

Velocity:

$$v(t)=r'(t)=<f'(t),g'(t),h'(t)>$$

Speed: Magnitude of velocity. Also related to the arc length:

$$\|v(t)\|=s'(t)$$

$$s=\int_{t_0}^{t_1}\|v(t)\|$$

Acceleration:

$$a(t)=r''(t)=<f''(t),g''(t),h''(t)>$$

Since $a(t)$ and $v(t)$ will always be at 90 degree angles from each other it makes sense that:

$$a(t)\cdot v(t)=0$$

Centripetal Acceleration: If the acceleration vector is antiparallel to the position vector at all times t

$$x(t) = tv_0\cos(\theta)$$

$$y(t) = -\frac{1}{2}gt^2 + tv_0\sin(\theta) + s_0$$

Section 9.3 – Curvature. Components of Acceleration

Unit Tangent (Tangential Component of Acceleration): A unit vector that is tangent to the curve.

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

Curvature (κ): The rate at which the unit vector T changes direction with respect to the length.

$$\kappa = \left\|\frac{dT}{ds}\right\| = \left\|\frac{T'(t)}{r'(t)}\right\|$$

The curvature of a circle with a radius a is $\frac{1}{a}$

Principle Normal (Normal Component of Acceleration):

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

Binormal Vector:

$$B(t) = T(t) \times N(t)$$

Makes it so the point can be seen as a moveable coordinate system since they're all perpendicular to each other: TNB-frame.

$$a_T = \frac{r'(t) \cdot r''(t)}{\|r'(t)\|}$$

$$a_N = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|}$$

$$\kappa(\mathrm{t}) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$$

Section 9.4 – Partial Derivatives

Functions of two variables:

$$z = f(x, y)$$

Where x and y are independent variables and z is dependent.

Level Curves: Curves defined by $f(x,y) = c$ see figure 9.4.2

Function of three or More Variables:

$$w = F(x,y,z)$$

Level Surfaces: make the equation equal to a constant c and then isolate the z so it becomes a two variable function.

Partial Derivatives:

To compute $\frac{dz}{dx}$ use the laws of ordinary differentiation while treating y as a constant

To compute $\frac{dz}{dy}$ use the laws of ordinary differentiation while treating x as a constant

Second Order Partial Derivatives:

$$\frac{d^2z}{dx^2} = \frac{d}{dx}\left(\frac{dz}{dx}\right)$$

Note the order of differentiations:

$$\frac{d^2z}{dxdy} = \frac{d}{dx}\left(\frac{dz}{dy}\right)$$

Chain Rule:

If $z = f(u,v)$ is differentiable and $u = g(x,y)$ and $v = h(x,y)$ have continuous first partial derivatives, then

$$\frac{dz}{dx} = \frac{dz}{du}\frac{du}{dx} + \frac{dz}{dv}\frac{dv}{dx}$$

Also tree diagrams here.

Section 9.5 – Directional Derivatives

DEL Operator (Vector Differential Operator):

$$\nabla = < \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} >$$

Gradient: When the DEL Operator is applied to a function

$$\nabla F(x,y,z) = \frac{dF}{dx}i + \frac{dF}{dy}j + \frac{dF}{dz}k$$

Directional Derivative:

$$D_u f(x, y, z) = \nabla F(x, y, z) \cdot u$$

Where u is a unit vector in the desired direction

The *Maximum Value* of the Directional derivative is $\|\nabla f\|$ and it occurs when u has the same direction as ∇f (when $\cos(\emptyset) = 1$).

The *Minimum Value* of the Directional derivative is $-\|\nabla f\|$ and it occurs when u has the opposite direction from ∇f (when $\cos(\emptyset) = -1$).

The gradient vector ∇f points in the direction in which f increases most rapidly, whereas $-\nabla f$ points in the direction of most rapid decrease of f.

Section 9.6 – Tangent Planes and Normal Lines

∇F is normal (perpendicular) to the level surface at point P

Tangent Plane: The plane at P is the plane through P that is normal to ∇F evaluated at P

Equation of a Tangent Plane:

Let $P(x_0, y_0, z_0)$ be a point on the graph of $F(x, y, z) = c$, where ∇F is not 0. Then an equation of the tangent plane at P is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Section 9.7 – Divergence and Curl

Vector Fields: Vector functions of two or more variables.

$$F(x, y) = P(x, y)i + Q(x, y)j$$

$$F(x, y, z) = P(x, y, z)i + Q(x, y, z)j + R(x, y, z)k$$

Curl: The curl of a vector field $F = Pi + Qj + Rk$ is the vector field:

$$curl\ F = \nabla \times F = \begin{vmatrix} i & j & k \\ {}^{d}/_{dx} & {}^{d}/_{dy} & {}^{d}/_{dz} \\ P & Q & R \end{vmatrix}$$

If the curl is 0 then the field is called *irrotational*.

Divergence: The divergence of a vector field $F = Pi + Qj + Rk$ is the scalar field:

$$div\ F = \nabla \cdot F = \frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz}$$

If the divergence is 0 then the field is said to be *incompressible*.

Section 9.8 – Line Integrals

Suppose C is a curve parameterized by $x = f(t), y = g(t), a \leq t \leq b$ and A and B are the points $(f(a), g(a))$ and $(f(b), g(b))$ respectively:

- C is a smooth curve if f' and g' are continuous over the closed interval $[a, b]$ and not simultaneously zero on the open interval (a, b)
- C is a piecewise smooth function if it consists of a finite number of smooth curves joined end to end
- C is a closed curve if $A = B$
- C is a simple closed curve if $A = B$ and the curve does not cross itself.

Line integrals are basically a lot of plugging and chugging. Get f,g and h from parametric equations

$$\int_C G(x,y)\,dx = \int_a^b G((f(t), g(t))f'(t)\,dt$$

$$\int_C G(x,y)\,dy = \int_a^b G((f(t), g(t))g'(t)\,dt$$

$$\int_C G(x,y)\,ds = \int_a^b G((f(t), g(t))\sqrt{[f'(t)]^2 + [g'(t)]^2}\ dt$$

Differential of Arc Length:

$$ds = \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$$

This is what they look like in problems:

$$\int_C Pdx + Qdy$$

For closed curves break it up into different segments

$$\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$$

3-Space

$$\int_C G(x,y,z)\,ds = \int_a^b G(f(t), g(t), h(t))\,\sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt$$

Work:

$$W = \int_C P(x,y)\,dx + Q(x,y)\,dy = \int_C F \cdot dr = \int_C comp_T F\,ds$$

Where dr is the derivative of the vector function r

The work done by a force F along a curve C is due entirely to the tangential component of F

Circulation: A line integral of a vector field F around a simple closed curve.

Section 9.9 – Independence of Path

Conservative Vector Field: A vector function F in 2 or 3-Space written as the gradient of a scalar function ∅. The function ∅ is called the potential function for F. I guess it's just any scalar function.

In other words, F is conservative if there exists a function ∅ such that $F = \nabla\emptyset$. Also called gradient vector field.

Fundamental Theorem of Line Integrals: Suppose C is a path in an open region R of the xy-plane and is defined by $r(t) = x(t)i + y(t)j, a \leq t \leq b$. If $F(x,y) = P(x,y)i + Q(x,y)j$ is a conservative vector field in R and ∅ is a potential function for F then

$$\int_C F \cdot dr = \int_C \nabla\emptyset \cdot dr = \emptyset(B) - \emptyset(A)$$

Where $A = (x(a), y(a))$ and $B = (x(b), y(b))$

Steps to Solve with Path independence:

1. Check to see if the vector field F is conservative ($curl\,F = 0$)
    - In 2-Space $P_y == Q_x$ then $curl\,F = 0$
2. Find ∅
    - Assume $< P, Q > = < \emptyset_x, \emptyset_y >$
    - Set up system of equations (like solving an EDE from 213 kind of)
3. Apply the Fundamental Theorem of Line Integrals (FTLI)
    - As such:

$$\int_{(x_0,y_0)}^{(x_1,y_1)} F \cdot dr = \emptyset(x_1, y_1) - \emptyset(x_0, y_0)$$