1. (**25 marks**) Circle the correct answer.

(a) The general solution of the differential equation $xy' + 3y = 3$ is
(A) $y = x^2 + cx^5$ (B) $y = x^5 + cx^2$
(C) $y = 1 + c/x^3$ (D) $y = x + c/x^3$

(b) The general solution of the differential equation $xy' = x + y$ is
(A) $x^2 + y^2 = cx$ (B) $y = cx + x\ln x$
(C) $y = c\sqrt{x} + x$ (D) $y^2 = x^2 + 2cxy$

(c) The general solution of the differential equation $2y'' - 3y' - 20y = 0$ is
(A) $y = c_1x + c_2x^2$ (B) $y = c_1e^x + c_2e^{x/2}$
(C) $y = c_1e^{-5x/2} + c_2e^{4x}$ (D) $y = c_1\cos x + c_2\sin x$

(d) An integrating factor $\mu(x, y)$ of the differential equation
$(2x\cos x)y' - xy\sin x + 2y\cos x = 0$ is
(A) x^2y^2 (B) $(x + y)^{-2}$
(C) $(x + y)^2$ (D) $1/\sqrt{\cos x}$

(e) The Wronskian $W(1, e^{2x}, e^{-x})$ is equal to
(A) $6e^x$ (B) 2
(C) $-2e^{3x}$ (D) $-2e^x$

(f) Given $y_{p_1} = e^{3x}$ and $y_{p_2} = -e^{-3x}$ are, respectively, particular solutions of

$$\begin{aligned} y'' - 3y' + 2y &= 2e^{3x} \\ y'' - 3y' + 2y &= -20e^{-3x} \end{aligned}$$

then a particular solution of $y'' - 3y' + 2y = 10(e^{3x} - 10e^{-3x})$ is
(A) $y_p = 5e^{3x} - \frac{1}{2}e^{-3x}$ (B) $y_p = 5e^{3x} + \frac{1}{2}e^{-3x}$
(C) $y_p = \frac{1}{2}e^{3x} + 5e^{5x}$ (D) $y_p = 5e^{3x} - 5e^{-3x}$

(g) If 100 mg of a radioactive substance decays to 80 mg in 5 hours, what is its half-life?
(A) 15.53 hours (B) 22.47 hours
(C) 1.72 hours (D) 12.42 hours

(h) A cylindrical tank initially contains water to a depth of 9 ft, and a bottom plug is removed at time $t = 0$ (hours). After 1 h the depth of the water has dropped to 4 ft. How long does it take for all the water to drain from the tank?
(A) 4 hours (B) 3 hours

(C) 6 hours (D) none of the above

(i) A 60-gal tank initially contains 20 gal of pure water. Salt-water solution containing 0.5 lb of salt for each gallon of water begins entering the tank at a rate of 4 gal/min. Simultaneously, the salt-water solution is pumped out of the tank at a rate of 2 gal/min. The amount of salt (in lb) in the tank after 15 min is
(A) 25 lb (B) 30 lb
(C) 21 lb (D) 15 lb

(j) A cool beer at 40 ^{0}F is placed into a warm room at 70 ^{0}F. Suppose 10 minutes later, the temperature of the beer is 48 ^{0}F. Using Newton's cooling law, the temperature of the beer after 25 minutes it is placed in the room is
(A) 61.28 ^{0}F (B) 50.28 ^{0}F
(C) 56.18 ^{0}F (D) 53.86 ^{0}F

2. [30 marks]
Solve the following initial value problems. When it is possible, express the solutions explicitly in terms of x.
a) [6 marks]
$$\frac{dy}{dx} + 3y = 2e^{-x}, \;\; y(0) = 1$$
b) [6 marks]
$$xy\frac{dy}{dx} = y^2 + 2x^2, \;\; y(1) = 1$$
c) [6 marks]
$$2xy^2 + 3x^2 + x + (2x^2y + y)\frac{dy}{dx} = 0, \;\; y(1) = 1$$
d) [6 marks]
$$\frac{dy}{dx} + 6y = 5xy^2, \;\; y(0) = 1$$
e) [6 marks] $y'' - 4y = 2x - 1$, $y(0) = 1$, $y'(0) = 3$

a) The DE is a FOLDE.

$$I(x) = e^{3x}$$

$$\therefore \quad e^{3x}\frac{dy}{dx} + 3e^{3x}y = 2e^{-x}e^{3x}$$

$$\frac{d}{dx}\left(e^{3x}y\right) = 2e^{2x}$$

$$\therefore \quad e^{3x}y = e^{2x} + C$$

$$1 = 1 + C$$

$$\therefore \; C = 0$$

$$\therefore \quad y = e^{-x} \quad \text{//Ans.}$$

b) Rewriting the DE as
$$\frac{dy}{dx} = \frac{y^2 + 2x^2}{xy}$$

Since the RHS is a quotient of two homogeneous functions of the same degree, we use $y = vx$. Then
$$\frac{dv}{dx}x + v = \frac{v^2x^2 + 2x^2}{vx^2}$$
$$= \frac{x^2(v^2+2)}{x^2 v}$$
$$= \frac{v^2+2}{v}$$
$$\frac{dv}{dx}x + v = v + \frac{2}{v}$$
$$\frac{dv}{dx}x = \frac{2}{v}$$
$$\therefore \quad v\,dv = 2\frac{dx}{x} \qquad \text{separable}$$

Integrating
$$\frac{1}{2}v^2 = 2\ln x + c_1$$
$$v^2 = 4\ln x + C$$
$$\frac{y^2}{x^2} = 4\ln x + C$$

Using the IC,
$$1 = C$$

$\therefore$ The solution is $y^2 = x^2(4\ln x + 1) \Rightarrow y = x\sqrt{4\ln x + 1}$ //Ans.

OR You can write the DE as

$$\frac{dy}{dx} - \frac{1}{x}y = 2xy^{-1}$$ which is a Bernoulli DE with $n=-1$. Multiplying both sides by y

$$y\frac{dy}{dx} - \frac{1}{x}y^2 = 2x$$

Let $v = y^2$. Then

$$\frac{dv}{dx} = 2y\frac{dy}{dx} \Rightarrow y\frac{dy}{dx} = \frac{1}{2}\frac{dv}{dx}$$

Substituting,

$$\frac{1}{2}\frac{dv}{dx} - \frac{1}{x}v = 2x$$

$$\frac{dv}{dx} - \frac{2}{x}v = 4x \quad \text{a FOLDE.}$$

$$I(x) = e^{-\int \frac{2}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$$

$$\therefore \quad x^{-2}\frac{dv}{dx} - 2x^{-3}v = 4x(x^{-2})$$

$$\frac{d}{dx}\left(\frac{v}{x^2}\right) = \frac{4}{x}$$

$$\therefore \quad \frac{v}{x^2} = 4\ln x + C$$

$$\therefore \quad y^2 = x^2(4\ln x + C)$$

Using the IC, $y(1)=1$, we obtain

$$1 = C$$

$$\therefore \quad y^2 = x^2(4\ln x + 1) \quad // \text{Ans.}$$

$$\Rightarrow y = x\sqrt{4\ln x + 1} \quad // \text{Ans.}$$

3. Rewriting the DE in differential form

$$\underbrace{(2xy^2 + 3x^2 + x)}_{M}dx + \underbrace{(2x^2y + y)}_{N}dy = 0$$

$$\left.\begin{array}{l} M_y = 4xy \\ N_x = 4xy \end{array}\right\} \Rightarrow M_y = N_x \Rightarrow \text{the DE is exact.}$$

Hence, there exists an F such that

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N$$

Integrating the first wrt x and the second wrt y,

$$F(x,y) = x^2y^2 + x^3 + \frac{1}{2}x^2 + g_1(y) \quad (A)$$

$$F(x,y) = x^2y^2 + \frac{1}{2}y^2 + g_2(x) \quad (B)$$

If (A) is compatible with (B), we must have

$$g_1(y) = \frac{1}{2}y^2 \quad \text{and} \quad g_1(x) = x^3 + \frac{1}{2}x^2$$

$\therefore \quad F(x,y) = x^2y^2 + x^3 + \frac{1}{2}x^2 + \frac{1}{2}y^2$

$\therefore$ The general solution is

$$x^2y^2 + x^3 + \frac{1}{2}x^2 + \frac{1}{2}y^2 = C$$

Using the IC, $y(1) = 1$, we obtain

$$C = 3$$

$\therefore$ The solution is

$$x^2y^2 + x^3 + \frac{1}{2}x^2 + \frac{1}{2}y^2 = 3 \quad // \text{Ans.}$$

d) This is a Bernoulli DE with $n = 2$. Dividing both sides of the equation by y^2.

$$\frac{1}{y^2}\frac{dy}{dx} + 6y^{-1} = 5x$$

Let $v = y^{-1}$. Then

$$\frac{dv}{dx} = -y^{-2}\frac{dy}{dx} \Rightarrow \frac{1}{y^2}\frac{dy}{dx} = -\frac{dv}{dx}$$

Substituting,

$$-\frac{dv}{dx} + 6v = 5x$$

$\therefore \quad \frac{dv}{dx} - 6v = -5x \quad$ a FOLDE.

$$I(x) = e^{-6x}$$

$\therefore \quad \frac{dv}{dx}e^{-6x} - 6e^{-6x}v = -5xe^{-6x}$

$$\frac{d}{dx}\left(\frac{v}{e^{6x}}\right) = -5xe^{-6x}$$

$\therefore \quad \frac{v}{e^{6x}} = -5\int xe^{-6x}\,dx$

$$= -5\left(-\frac{1}{6}xe^{-6x} - \frac{1}{36}e^{-6x}\right) + C$$

$$\begin{array}{c|c} u & dv \\ \hline x & e^{-6x} \\ 1 & \frac{e^{-6x}}{-6} \quad + \\ 0 & \frac{e^{-6x}}{36} \quad - \end{array}$$

$\therefore \quad v = \frac{5}{6}\left(x + \frac{1}{6}\right) + Ce^{6x}$

$$\frac{1}{y} = \frac{5}{6}\left(x + \frac{1}{6}\right) + Ce^{6x}$$

Using the IC, $y(0) = 1$, we obtain

$$1 = \frac{5}{36} + C \Rightarrow C = \frac{31}{36}$$

$\therefore \quad \frac{1}{y} = \frac{5}{6}\left(x + \frac{1}{6}\right) + \frac{31}{36}$

$$= \frac{5}{6}x + 1$$

$\therefore \quad y = \frac{6}{6 + 5x} \quad // \text{Ans.}$

e) $y'' - 4y = 2x - 1$, $y(0) = 1$, $y'(0) = 3$

First we solve the associated homogeneous eqn to obtain y_c.

$$y'' - 4y = 0$$

CE: $m^2 - 4 = 0 \Rightarrow m^2 = 4 \Rightarrow m = \pm 2$

$\therefore\ y_1 = e^{2x}$, $y_2 = e^{-2x}$

$\therefore\ y_c = c_1 y_1 + c_2 y_2 = c_1 e^{2x} + c_2 e^{-2x}$ //

Next, we find y_p. Since $f(x) = 2x - 1$, we may use

$$y_p = Ax + B$$

then $y_p' = A$

$y_p'' = 0$

Substituting these into the DE, we obtain

$$-4(Ax + B) = 2x - 1$$

$$-4Ax - 4B = 2x - 1$$

Equating the coefficients,

$$\begin{cases} -4A = 2 \\ -4B = -1 \end{cases} \Rightarrow \begin{array}{l} A = -\frac{1}{2} \\ B = \frac{1}{4} \end{array}$$

$\therefore\ y_p = -\frac{1}{2}x + \frac{1}{4}$ //

$\therefore\ y = y_c + y_p$

$= c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{2}x + \frac{1}{4}$

$\Rightarrow\ y' = 2c_1 e^{2x} - 2c_2 e^{-2x} - \frac{1}{2}$

Using the IC's, $y(0) = 1$ and $y'(0) = 3$, we obtain

$$\begin{cases} c_1 + c_2 + \frac{1}{4} = 1 \\ 2c_1 - 2c_2 - \frac{1}{2} = 3 \end{cases} \Rightarrow \begin{array}{l} c_1 + c_2 = \frac{3}{4} \\ 2c_1 - 2c_2 = \frac{7}{2} \end{array} \Rightarrow \begin{array}{l} c_1 = \frac{5}{4} \\ c_2 = -\frac{1}{2} \end{array}$$

$\therefore$ The solution to the IVP is

$$y = \frac{5}{4}e^{2x} - \frac{1}{2}e^{-2x} - \frac{1}{2}x + \frac{1}{4} \quad \text{// Ans.}$$

3. **[9 marks]** Solve the Initial Value Problem.

$$y'' + 4y = 2\sin 3x + \cos x, \ \ y(0) = 1, \ y'(0) = 0$$

$$= 2\sin 3x + \cos 3x$$

First we solve the associated homogeneous DE

$$y'' + 4y = 0$$

CE : $m^2 + 4 = 0$

$\therefore \ m = \pm 2i$

$\therefore \ y_c = c_1 \cos 2x + c_2 \sin 2x$ //

Use the method of undetermined coefficients, we let

$$y_p = A\sin 3x + B\cos 3x + C\cos x + D\sin x$$

$\Rightarrow y_p' = 3A\cos 3x - 3B\sin 3x - C\sin x + D\cos x$

$y_p'' = -9A\sin 3x - 9B\cos 3x - C\cos x - D\sin x$

Subst these into the DE,

$$\underline{-9A\sin 3x} - \underline{\underline{9B\cos 3x}} - C\cos x - D\sin x + \underline{4A\sin 3x} + \underline{\underline{4B\cos 3x}} + 4C\cos x + 4D\sin x = 2\sin 3x + \cos x$$

$$-5A\sin 3x - 5B\cos 3x + 3C\cos x + 3D\sin x = 2\sin 3x + \cos x$$

Equating the corresponding coefficients,

$$\begin{cases} -5A = 2 \\ -5B = 0 \\ 3C = 1 \\ 3D = 0 \end{cases} \Rightarrow \begin{array}{l} A = -\frac{2}{5} \\ B = 0 \\ C = \frac{1}{3} \\ D = 0 \end{array}$$

$\therefore \ y_p = -\frac{2}{5}\sin 3x + \frac{1}{3}\cos x$ //

$\therefore \ y = y_c + y_p$

$= c_1 \cos 2x + c_2 \sin 2x - \frac{2}{5}\sin 3x + \frac{1}{3}\cos x$

$\Rightarrow y' = -2c_1 \sin 2x + 2c_2 \cos 2x - \frac{6}{5}\cos 3x - \frac{1}{3}\sin x$

Using the IC's, we obtain

$y(0) = 1 = c_1 + \frac{1}{3} \Rightarrow c_1 = \frac{2}{3}$

$y'(0) = 0 = 2c_2 - \frac{6}{5} \Rightarrow c_2 = \frac{3}{5}$

$\therefore$ The Solution is

$$y = \frac{2}{3}\cos 2x + \frac{3}{5}\sin 2x - \frac{2}{5}\sin 3x + \frac{1}{3}\cos x \quad \text{//Ans.}$$

If you use $f(x) = 2\sin 3x + \cos 3x$

then

$$y_p = A\sin 3x + B\cos 3x$$

$$\Rightarrow \quad y_p' = 3A\cos 3x - 3B\sin 3x$$

$$y_p'' = -9A\sin 3x - 9B\cos 3x$$

Subst these into the DE

$$\underline{-9A\sin 3x} \; \underline{\underline{-9B\cos 3x}} + \underline{4A\sin 3x} + \underline{\underline{4B\cos 3x}} = 2\sin 3x + \cos 3x$$

$$-5A\sin 3x - 5B\cos 3x = 2\sin 3x + \cos 3x$$

Equating the corresponding coefficients,

$$\begin{cases} -5A = 2 \\ -5B = 1 \end{cases} \Rightarrow \quad \begin{array}{l} A = -\frac{2}{5} \\ B = -\frac{1}{5} \end{array}$$

$$\therefore \quad y_p = -\frac{2}{5}\sin 3x - \frac{1}{5}\cos 3x \quad //$$

$$\therefore \quad y = y_c + y_p$$

$$= c_1\cos 2x + c_2\sin 2x - \frac{2}{5}\sin 3x - \frac{1}{5}\cos 3x$$

$$\Rightarrow y' = -2c_1\sin 2x + 2c_2\cos 2x - \frac{6}{5}\cos 3x + \frac{3}{5}\sin 3x$$

Using the IC's,

$$\begin{array}{l} y(0) = 1 = c_1 - \frac{1}{5} \\ y'(0) = 0 = 2c_2 - \frac{6}{5} \end{array} \Rightarrow \begin{array}{l} c_1 = \frac{6}{5} \\ c_2 = \frac{3}{5} \end{array}$$

$\therefore$ The solution is

$$y = \frac{6}{5}\cos 2x + \frac{3}{5}\sin 2x - \frac{2}{5}\sin 3x - \frac{1}{5}\cos 3x \quad //$$

4. **[8 marks]** Find a particular solution y_p of the given differential equation

$$y''' - 2y'' + y' = 1 - 6e^x + 25e^{5x}$$

From $f(x)$, we may have an initial guess for y_P

$$y_P = A + Be^x + Ce^{5x}$$

Now we consider y_c which is the general solution of the homogeneous Equation

$$y''' - 2y'' + y' = 0$$

CE: $m^3 - 2m^2 + m = 0$

$$m(m^2 - 2m + 1) = 0$$

$$m(m-1)^2 = 0 \Rightarrow m = 0, 1, 1$$

$\therefore$ $y_1 = 1$, $y_2 = e^x$, $y_3 = xe^x$

$\therefore$ $y_c = c_1 + c_2 e^x + c_3 xe^x$

Since there is a duplication between y_c and the initial guess for y_P, we must modify the guess.

$$y_P = Ax + Bx^2e^x + Ce^{5x}$$

$$\Rightarrow y_P' = 2Bxe^x + Bx^2e^x + 5Ce^{5x} + A$$

$$= B(2x + x^2)e^x + 5Ce^{5x} + A$$

$$y_P'' = B(2 + 2x)e^x + B(2x + x^2)e^x + 25Ce^{5x}$$

$$= B(2 + 4x + x^2)e^x + 25Ce^{5x}$$

$$y''' = B(4 + 2x)e^x + B(2 + 4x + x^2)e^x + 125Ce^{5x}$$

$$= B(6 + 6x + x^2)e^x + 125Ce^{5x}$$

Subst these into the DE,

$$B(6 + 6x + x^2)e^x + 125Ce^{5x} - 2B(2 + 4x + x^2)e^x - 50Ce^{5x}$$

$$+ B(2x + x^2)e^x + 5Ce^{5x} + A = 1 - 6e^x + 25e^{5x}$$

$\therefore$ $2Be^x + 80Ce^{5x} + A = 1 - 6e^x + 25e^{5x}$

Equating the corresponding coefficients,

$$\begin{cases} A = 1 \\ 2B = -6 \\ 80C = 25 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = -3 \\ C = \frac{5}{16} \end{cases}$$

$\therefore$ $y_P = x - 3x^2e^x + \frac{5}{16}e^{5x}$ //Ans.

5. **[8 marks]** Use the method of Variation of Parameters to find a particular solution of the following differential equation.

$$y'' + 4y = \sec 2x$$

First we solve the associated homogeneous DE

$$y'' + 4y = 0$$

CE: $m^2 + 4 = 0 \Rightarrow m^2 = -4 \Rightarrow m^2 = 4i^2$

$\Rightarrow m = \pm 2i$

$\therefore \quad y_1 = \cos 2x \,, \quad y_2 = \sin 2x$

$\therefore \quad y_c = c_1 \cos 2x + c_2 \sin 2x$

Using VOPs, we let

$$y_p = u_1 \cos 2x + u_2 \sin 2x$$

where u_1 and u_2 satisfy

$$\begin{cases} u_1' \cos 2x + u_2' \sin 2x = 0 \\ -2u_1' \sin 2x + 2u_2' \cos 2x = \sec 2x \end{cases}$$

Using Cramer's rule

$$u_1' = \frac{\begin{vmatrix} 0 & \sin 2x \\ \sec 2x & 2\cos 2x \end{vmatrix}}{\begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}} = \frac{-\sec 2x \sin 2x}{2\cos^2 2x + 2\sin^2 2x}$$

$$= -\frac{\sec 2x \sin 2x}{2\underbrace{(\cos^2 2x + \sin^2 2x)}_{1}} = -\frac{1}{2}\sec 2x \sin 2x$$

$$u_2' = \frac{\begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & \sec 2x \end{vmatrix}}{\begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}} = \frac{\cos 2x \sec 2x}{2} = \frac{1}{2}$$

$\therefore \quad u_1 = -\frac{1}{2}\int \sec 2x \sin 2x \, dx$

$= -\frac{1}{2}\int \frac{\sin 2x}{\cos 2x}\,dx = \frac{1}{2}\cdot\frac{1}{2}\int \frac{-2\sin 2x}{\cos 2x}\,dx$

$= \frac{1}{4}\ln|\cos 2x|$

$u_2 = \int \frac{1}{2}\,dx = \frac{1}{2}x$

$\therefore \quad y_p = \frac{1}{4}\cos 2x \ln|\cos 2x| + \frac{1}{2}x \sin 2x \quad$ // Ans.

6. [**8 marks**] Suppose that the population $P(t)$ of a country satisfies the logistic differential equation
$$\frac{dP}{dt} = kP(200-P)$$
with k is a constant. Its population in 1940 was 100 million and was then growing with a rate of 1 million per year i.e., $P'(0) = 1$ million/yr. Solve the above initial value problem and predict this country's population for the year 2000.

Rewriting the DE as
$$\frac{dP}{P(200-P)} = k\,dt \quad \text{which is separable}$$
Integrating
$$\int \frac{dP}{P(200-P)} = k\int dt \qquad (*)$$
Decomposing the integrand on the LHS to partial fractions
$$\frac{1}{P(200-P)} = \frac{A}{P} + \frac{B}{200-P}$$
where
$$A = \left.\frac{1}{200-P}\right|_{P=0} = \frac{1}{200}$$
$$B = \left.\frac{1}{P}\right|_{P=200} = \frac{1}{200}$$
$\therefore$ $(*)$ becomes
$$\frac{1}{200}\Big[\ln P - \ln(200-P)\Big] = kt + c_1$$
$$\therefore \quad \ln\frac{P}{200-P} = 200kt + c_2$$
$$\therefore \quad \frac{P}{200-P} = Ce^{200kt} \quad //$$
Using the IC, $P(0) = 100$, we obtain
$$\frac{100}{200-100} = C \Rightarrow C = 1$$
$$\therefore \quad \frac{P}{200-P} = e^{200kt}$$
$$\therefore \quad P = 200e^{200kt} - Pe^{200kt}$$
$$(1+e^{200kt})P = 200e^{200kt}$$
Differentiating wrt t and using $P'(0) = 1$,
$$200ke^{200kt}P + (1+e^{200kt})P' = 40000ke^{200kt}$$
$$200k(100) + 2(1) = 40000k$$
$$\therefore \quad k = \frac{2}{20000} = \frac{1}{10000} \quad //$$
$$\therefore \quad P(t) = \frac{200e^{200kt}}{1+e^{200kt}} = \frac{200}{1+e^{-200kt}} = \frac{200}{1+e^{-0.02t}}$$
$\therefore$ At the year 2000, $P(60) \approx 153.705$ million //Ans.

7. [**12 marks**] Solve the Cauchy-Euler differential equation

$$x^2y'' - 2xy' + 2y = \ln x \qquad (1)$$

subject to the initial conditions $y(1) = 0$, $y'(1) = 1$.

First we solve the associated homogeneous DE of (1), that is,

$$x^2 y'' - 2xy' + 2y = 0$$

Let $y = x^m$. Then

$$y' = m x^{m-1}$$
$$y'' = m(m-1)x^{m-2}$$

Subst these into (2),

$$m(m-1)x^m - 2mx^m + 2x^m = 0$$
$$x^m\left[m(m-1) - 2m + 2\right] = 0$$

Since $x \neq 0$, we obtain

$$m(m-1) - 2m + 2 = 0$$
$$m^2 - 3m + 2 = 0$$
$$(m-1)(m-2) = 0 \Rightarrow m_1 = 1,\ m_2 = 2$$

$\therefore\ y_1 = x,\ y_2 = x^2$

$\therefore\ y_c = c_1y_1 + c_2y_2$
$= c_1x + c_2x^2$

To obtain y_p we use VOP. Let

$$y_p = u_1x + u_2x^2$$

where u_1 & u_2 satisfying

$$\begin{cases} u_1'x + u_2'x^2 = 0 \\ u_1' + 2u_2'x = \dfrac{\ln x}{x^2} \end{cases}$$

Using Cramer's rule

$$u_1' = \frac{\begin{vmatrix} 0 & x^2 \\ \frac{\ln x}{x^2} & 2x \end{vmatrix}}{\begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}} = \frac{-x^2\frac{\ln x}{x^2}}{2x^2 - x^2} = -\frac{\ln x}{x^2}$$

$$u_2' = \frac{\begin{vmatrix} x & 0 \\ 1 & \frac{\ln x}{x^2} \end{vmatrix}}{\begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}} = \frac{x\frac{\ln x}{x^2}}{x^2} = \frac{\ln x}{x^3}$$

$$\therefore\ u_1 = -\int \frac{\ln x}{x^2}\,dx$$

Using IBPs, let $u = \ln x$, $dv = \dfrac{dx}{x^2}$

$du = \dfrac{dx}{x}$, $v = -\dfrac{1}{x}$

then $u_1 = -\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} - \int \frac{dx}{x^2}$

$= \frac{\ln x}{x} + \frac{1}{x}$

$u_2 = \int \frac{\ln x}{x^3}\,dx$

Using IBPs, let

$u = \ln x \quad dv = \frac{dx}{x^3}$

$du = \frac{dx}{x} \quad v = -\frac{1}{2x^2}$

$\therefore \quad u_2 = \left[-\frac{\ln x}{2x^2} + \frac{1}{2}\int \frac{dx}{x^3} \right]$

$= -\frac{\ln x}{2x^2} - \frac{1}{4x^2}$

$\therefore \quad y_p = \left(\frac{\ln x + 1}{x}\right) x - \left(\frac{\ln x}{2x^2} + \frac{1}{4x^2}\right) x^2$

$= \ln x + 1 - \frac{1}{2}\ln x - \frac{1}{4}$

$= \frac{1}{2}\ln x + \frac{3}{4}$ //

$\therefore \quad y = y_c + y_p$

$= c_1 x + c_2 x^2 + \frac{1}{2}\ln x + \frac{3}{4}$

$\Rightarrow y' = c_1 + 2c_2 + \frac{1}{2x}$

Using the IC's,

$\begin{cases} 0 = c_1 + c_2 + \frac{3}{4} \\ 1 = c_1 + 2c_2 + \frac{1}{2} \end{cases} \Rightarrow \quad \begin{matrix} c_1 + c_2 = -\frac{3}{4} \\ c_1 + 2c_2 = \frac{1}{2} \end{matrix} \quad \Rightarrow \quad \begin{matrix} c_1 = -2 \\ c_2 = \frac{5}{4} \end{matrix}$

$\therefore \quad y = -2x + \frac{5}{4}x^2 + \frac{1}{2}\ln x + \frac{3}{4}$ // Ans.