Series solution formulas

- These notes are not meant to explain the underlying theory in detail, they are intended to just give a summary of the formulas you will be responsible for.
- There is nothing superfluous in these notes. Take them seriously.
- None of these formulas will be given to you. It may seem daunting to memorize "all of these formulas", but once you start you'll find shortcuts to memorizing them (for instance Cases II and III are very, very similar, and there is a nice pattern in the general recurrence relation).
- I think it is worth your time to rewrite each and every formula in the special case where $x_0 = 0$.

Consider the second-order, linear, non-constant-coefficient DE

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$$

A **singular point** is a value x_0 such that $a_0(x_0) = 0$. (Very often, but not always, $x_0 = 0$.) Put the DE in standard form:

$$y'' + p(x)y' + q(x)y = 0\,.$$

The singular point x_0 is **regular** if the functions $(x - x_0)p(x)$ and $(x - x_0)^2q(x)$ are analytic at x_0. The test that we will use for regularity is:

$$\lim_{x\to x_0}(x - x_0)p(x) = p_0$$
$$\lim_{x\to x_0}(x - x_0)^2q(x) = q_0$$

where both p_0 and q_0 are finite.

The **indicial equation** is

$$F(r) = r(r - 1) + p_0r + q_0 = 0\,,$$

and is used to determine the two possible values for r (the *exponents*), $r = r_1$, $r = r_2$.

The **general recurrence relation** is

$$a_n(r) = \frac{-[(n + r - 1)p_1 + q_1]a_{n-1} - [(n + r - 2)p_2 + q_2]a_{n-2} - \cdots - [rp_n + q_n]a_0}{F(n + r)}$$

where p_n, q_n are the coefficients of the series expansions for $(x - x_0)p(x)$ and $(x - x_0)^2q(x)$, respectively.

Case I. If $r_1 \neq r_2$ and $r_1 - r_2 \neq N$ (where N is an integer), then two independent solutions are

$$y_1(x) = |x - x_0|^{r_1} \sum_{n=0}^{\infty} a_n(r_1)(x - x_0)^n$$
$$y_2(x) = |x - x_0|^{r_2} \sum_{n=0}^{\infty} a_n(r_2)(x - x_0)^n$$

where x_0 is the regular singular point, r_1, r_2 are the roots of the indicial equation, the coefficients $a_n(r_1)$ are determined by the recurrence relation that results from plugging $r = r_1$ into the general recurrence relation, and the coefficients $a_n(r_2)$ are determined by the recurrence relation that results from plugging $r = r_2$ into the general recurrence relation.

Case II. If $r_1 = r_2$, then two independent solutions are

$$y_1(x) = |x - x_0|^{r_1} \sum_{n=0}^{\infty} a_n(r_1)(x - x_0)^n$$
$$y_2(x) = \ln|x - x_0| y_1(x) + |x - x_0|^{r_1} \sum_{n=1}^{\infty} b_n(r_1)(x - x_0)^n$$

where x_0 is the regular singular point, r_1 is the double root of the indicial equation, the coefficients $a_n(r_1)$ are determined by the recurrence relation that results from plugging $r = r_1$ into the general recurrence relation, and the coefficients $b_n(r_1)$ are determined by the recurrence relation that results from differentiating the general recurrence relation with respect to r and then plugging in $r = r_2$. **Note that for y_2 the summation starts at $n = 1$.**

Case III. If $r_1 - r_2 = N$ (where N is a *positive* integer, so take r_1 to be the larger root), then two independent solutions are

$$y_1(x) = |x - x_0|^{r_1} \sum_{n=0}^{\infty} a_n(r_1)(x - x_0)^n$$
$$y_2(x) = a \ln|x - x_0| y_1(x) + |x - x_0|^{r_2} \sum_{n=0}^{\infty} c_n(r_2)(x - x_0)^n$$

where x_0 is the regular singular point, r_1, r_2 are the roots of the indicial equation, the coefficients $a_n(r_1)$ are determined by the recurrence relation that results from plugging $r = r_1$ into the general recurrence relation, and the coefficients $c_n(r_2)$ are determined by complicated formulas that do not concern us. **Note that these forms are almost identical to the previous case** $r_1 = r_2$, the only differences being that the summation in y_2 starts at $n = 0$ (instead of at $n = 1$ in the previous case), and there is a leading coefficient, a, that could, in principle, be zero.

Note: Often, we are content to determine just a few nonzero terms of each solution. Then we proceed to find r_1 and r_2, and write down the appropriate form for the two solutions (Case I, II, or III). Then expand out a few terms for y_1 and y_2 [for example, if $x_0 = 0$ and $r_1 = r_2$, we would write $y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$ and $y_2(x) = y_1(x) \ln|x| + x^{r_1}(b_1 x + b_2 x^2 + b_3 x^3 + \cdots)$]. Then we would plug these directly into the DE, group terms according to powers of x, and determine a set of equations first for a_0, a_1, a_2, a_3, and then (using these) for b_1, b_2, b_3 (see the practice problems that are posted on WebCT).

The behaviour of solutions as $x \to x_0$. In each case, the behaviour of each solution $y_1(x), y_2(x)$ is determined by the *leading term*. So:

For **Case I**:

$$\begin{aligned} y_1(x) &\sim |x - x_0|^{r_1} \quad \text{as } x \to x_0 \\ y_2(x) &\sim |x - x_0|^{r_2} \quad \text{as } x \to x_0 \end{aligned}$$

For **Case II**:

$$\begin{aligned} y_1(x) &\sim |x - x_0|^{r_1} \quad \text{as } x \to x_0 \\ y_2(x) &\sim |x - x_0|^{r_1} \ln|x - x_0| + |x - x_0|^{r_1} \quad \text{as } x \to x_0 \end{aligned}$$

For **Case III**:

$$\begin{aligned} y_1(x) &\sim |x - x_0|^{r_1} \quad \text{as } x \to x_0 \\ y_2(x) &\sim a|x - x_0|^{r_1} \ln|x - x_0| + |x - x_0|^{r_2} \quad \text{as } x \to x_0 \end{aligned}$$

(**Note** that these formulas are easy to remember if you can remember the general forms for $y_1(x)$ and $y_2(x)$ that were given above; just imagine expanding the series out, take the *leading term* and don't worry about the constants.)

A case-by-case breakdown of what happens as $x \to x_0$ is given in the course notes ($r_1 > 0$ and $r_2 < 0$; $r_1 > r_2 > 0$; and so on and so on) but all you need to remember to solve a problem are the following facts:

- If $r \geq 0$, then $|x - x_0|^r$ is **bounded** as $x \to x_0$ (think of $x, x^{1/2}, x^2$, etc).
- If $r < 0$, then $|x - x_0|^r$ is **unbounded** as $x \to x_0$ (think of $1/x, 1/x^2$, etc).
- If $r > 0$, then $|x - x_0|^r \ln|x - x_0|$ is **bounded** as $x \to x_0$ (this can be proven using l'Hopital's Rule, for example; but you may just remember it as a fact).
- If $r \leq 0$, then $|x - x_0|^r \ln|x - x_0|$ is **unbounded** as $x \to x_0$ (think of $(\ln x)/x$, etc).