Assignment #4 - Solutions

Problem 1

P1a)

```
>> A = [1 3 -2 4 7; -3 2 1 3 5; 5 11 2 3 1; 7 2 4 -2 2; 3 5 7 1 7]

A =

     1     3    -2     4     7
    -3     2     1     3     5
     5    11     2     3     1
     7     2     4    -2     2
     3     5     7     1     7

>> B = [4; 2; 1; 5; 3]

B =

     4
     2
     1
     5
     3

>> X = A\B

X =

  156.4118
 -235.3529
  193.8824
  529.3529
 -168.0000
```

P1 b)

```
>> A(5,1) = 6

A =

     1     3    -2     4     7
    -3     2     1     3     5
     5    11     2     3     1
     7     2     4    -2     2
     6     5     7     1     7
```

P1 c)

```
>> det(A)

ans =

  -2.0110e+03

>> inv(A)

ans =

    0.1472   -0.5231   -0.0308   -0.3685    0.3362
   -0.1154    0.5992    0.1323    0.6131   -0.5067
   -0.0895   -0.2765   -0.0353   -0.4381    0.4172
    0.2332   -1.3018   -0.0691   -1.5196    1.1407
    0.0124    0.4828   -0.0229    0.5331   -0.3635
```

P1 d)

```
>> [L,U] = lu(A)

L =

    0.1429    0.2836   -0.6023    1.0000         0
   -0.4286    0.2985    0.7683   -0.0257    1.0000
    0.7143    1.0000         0         0         0
    1.0000         0         0         0         0
    0.8571    0.3433    1.0000         0         0


U =

    7.0000    2.0000    4.0000   -2.0000    2.0000
         0    9.5714   -0.8571    4.4286   -0.4286
         0         0    3.8657    1.1940    5.4328
         0         0         0    3.7490   10.1081
         0         0         0         0    2.0711
```

P1 e)

```
>> cond(A,2)     >> cond(A,inf)

ans =            ans =

   52.6261         110.8782
```

P2 a)

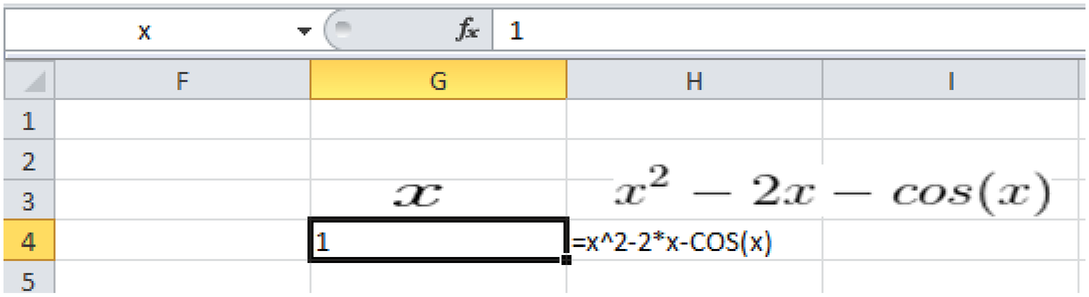
```
>> f = [1 -3 2 -4 -5 2]

f =

     1    -3     2    -4    -5     2

>> roots(f)

ans =

   2.9500 + 0.0000i
   0.2918 + 1.5351i
   0.2918 - 1.5351i
  -0.8575 + 0.0000i
   0.3238 + 0.0000i
```

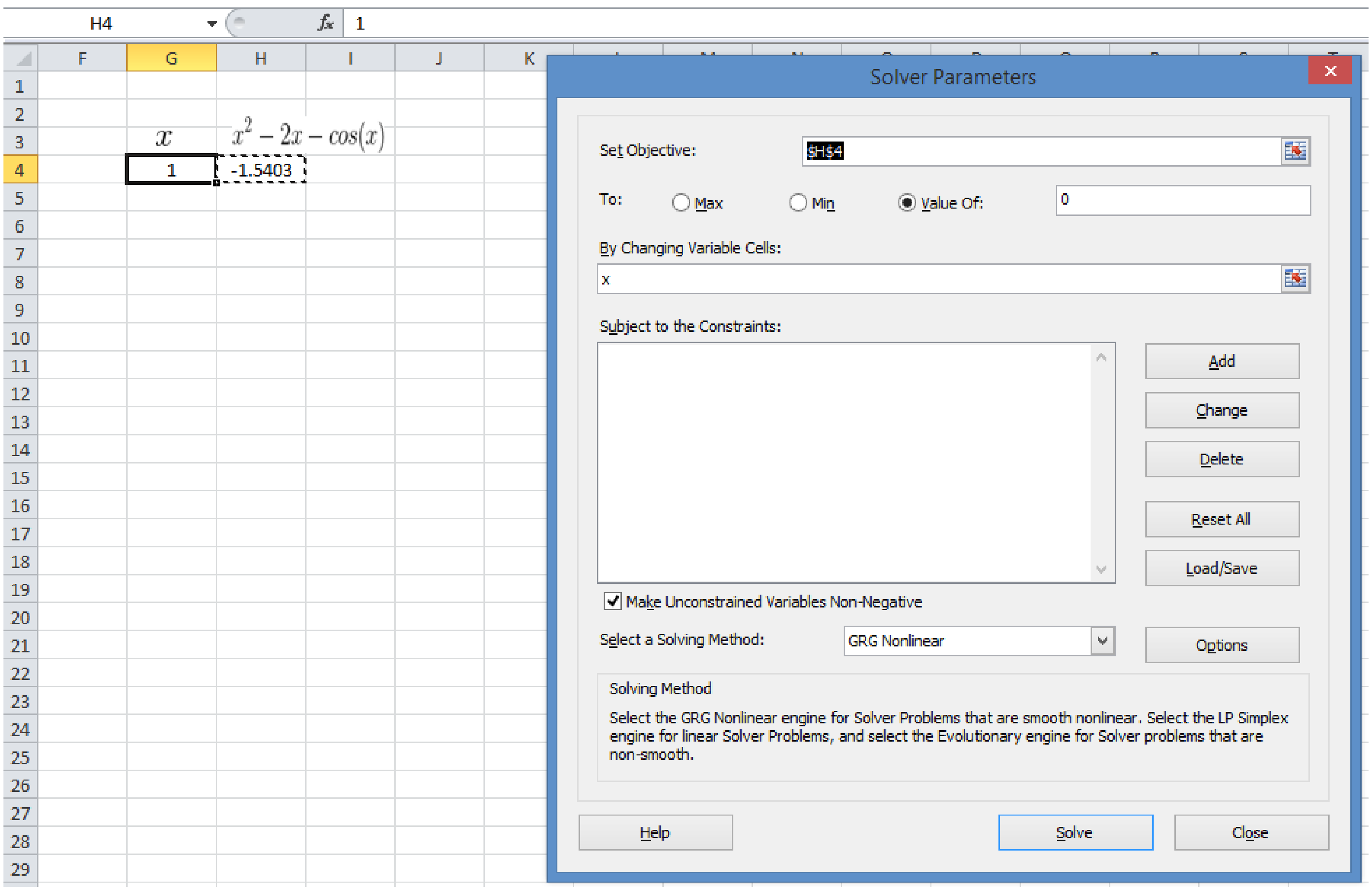
P2 b)

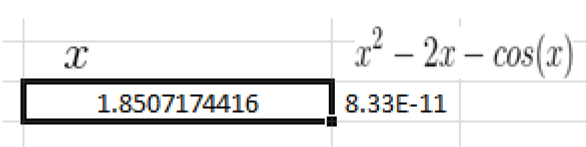
```
>> f = @(x) x.^5+exp(x)

f =

    @(x)x.^5+exp(x)

>> fzero(f,0)

ans =

   -0.8446
```

Problem 2

$$x^2 - 2x = \cos(x) \equiv \; x^2 - 2x - \cos(x) = 0$$







Problem 3

a- Jacobi's method: First we convert the system into an iterative form:

$$\begin{cases} a^{i+1} = \frac{27-b^i-c^i}{4} \\ b^{i+1} = \frac{36-a^i-c^i}{4} \\ c^{i+1} = \frac{60-a^i-2b^i}{5} \end{cases}$$

Now, we plug the initial values for $i = 0$ into the right hand side (RHS) of the system to find the results for first iteration for $i = 1$.

1st iteration:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}^0 = \begin{bmatrix} 2 \\ 7 \\ 7 \end{bmatrix}$$

$$\begin{cases} a^1 = \frac{27-7-7}{4} = 3.25 \\ b^1 = \frac{36-7-2}{4} = 6.75 \\ c^1 = \frac{60-2-2*7}{5} = 8.8 \end{cases}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}^1 = \begin{bmatrix} 3.25 \\ 6.75 \\ 8.8 \end{bmatrix}$$

2nd iteration:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}^1 = \begin{bmatrix} 3.25 \\ 6.75 \\ 8.8 \end{bmatrix}$$

$$\begin{cases} a^2 = \frac{27-6.75-8.8}{4} = 2.8625 \\ b^2 = \frac{36-8.8-3.25}{4} = 5.9875 \\ c^2 = \frac{60-3.25-2*6.75}{5} = 8.65 \end{cases}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}^2 = \begin{bmatrix} 2.8625 \\ 5.9875 \\ 8.65 \end{bmatrix}$$

b- True error and approximate relative errors are:

$$[True\ error] = [Exact\ ] - [Approximation]$$

$$e_{true} = \begin{bmatrix} e1 \\ e2 \\ e3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} - \begin{bmatrix} 2.8625 \\ 5.9875 \\ 8.65 \end{bmatrix} = \begin{bmatrix} -0.1375 \\ 0.0125 \\ 0.35 \end{bmatrix}$$

$$[Approximate\ relative\ error] = \frac{[Approximation]^2 - [Approximation]^1}{[Approximation]^2}$$

$$e_{app} = \begin{bmatrix} e1 \\ e2 \\ e3 \end{bmatrix} = \frac{\begin{bmatrix} 2.8625 \\ 5.9875 \\ 8.65 \end{bmatrix} - \begin{bmatrix} 3.25 \\ 6.75 \\ 8.8 \end{bmatrix}}{\begin{bmatrix} 2.8625 \\ 5.9875 \\ 8.65 \end{bmatrix}} = \begin{bmatrix} -0.135 \\ -0.1273 \\ -0.0173 \end{bmatrix}$$

The ∞ - norm of the error vector is the maximum of absolute values of the error vector:

$$||e_{true}||_\infty = \max(\text{abs}(e_{true})) = \max\left(abs \begin{bmatrix} -0.0.1375 \\ 0.0125 \\ 0.35 \end{bmatrix} \right) = 0.35$$

$$||e_{app}||_\infty = \max\left(\text{abs}\left(e_{app}\right)\right) = \max\left(abs \begin{bmatrix} -0.135 \\ -0.1273 \\ -0.0173 \end{bmatrix} \right) = 0.135$$

c- Gauss-Seidel method: First we convert the system into an iterative form:

$$\begin{cases} a^{i+1} = \frac{27 - b^i - c^i}{4} \\ b^{i+1} = \frac{36 - a^{i+1} - c^i}{4} \\ c^{i+1} = \frac{60 - a^{i+1} - 2b^{i+1}}{5} \end{cases}$$

Now, we plug the initial values for $i = 0$ into the right hand side (RHS) of the system to find the results for first iteration for $i = 1$. Note that we use the most recent value for second and third equations.

1st iteration:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}^0 = \begin{bmatrix} 2 \\ 7 \\ 7 \end{bmatrix}$$

$$\begin{cases} a^1 = \frac{27 - 7 - 7}{4} = 3.25 \\ b^1 = \frac{36 - 7 - 3.25}{4} = 6.4375 \\ c^1 = \frac{60 - 3.25 - 2 * 6.4375}{5} = 8.775 \end{cases}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}^1 = \begin{bmatrix} 3.25 \\ 6.4375 \\ 8.775 \end{bmatrix}$$

2nd iteration:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}^1 = \begin{bmatrix} 3.25 \\ 6.4375 \\ 8.775 \end{bmatrix}$$

$$\begin{cases} a^2 = \frac{27-6.4375-8.775}{4} = 2.947 \\ b^2 = \frac{36-8.775-2.947}{4} = 6.0695 \\ c^2 = \frac{60-2.947-2*6.0695}{5} = 8.9828 \end{cases}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}^2 = \begin{bmatrix} 2.947 \\ 6.0695 \\ 8.9828 \end{bmatrix}$$

d- True error and approximate relative errors are:

$$[True\ error] = [Exact\ ] - [Approximation]$$

$$e_{true} = \begin{bmatrix} e1 \\ e2 \\ e3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} - \begin{bmatrix} 2.947 \\ 6.0695 \\ 8.9828 \end{bmatrix} = \begin{bmatrix} 0.053 \\ -0.0695 \\ 0.0172 \end{bmatrix}$$

$$[Approximate\ relative\ error] = \frac{[Approximation]^2 - [Approximation]^1}{[Approximation]^2}$$

$$e_{app} = \begin{bmatrix} e1 \\ e2 \\ e3 \end{bmatrix} = \frac{\begin{bmatrix} 2.947 \\ 6.0695 \\ 8.9828 \end{bmatrix} - \begin{bmatrix} 3.25 \\ 6.4375 \\ 8.775 \end{bmatrix}}{\begin{bmatrix} 2.947 \\ 6.0695 \\ 8.9828 \end{bmatrix}} = \begin{bmatrix} -0.103 \\ -0.0606 \\ 0.0231 \end{bmatrix}$$

The ∞ - norm of the error vector is the maximum of absolute values of the error vector:

$$||e_{true}||_\infty = \max(\text{abs}(e_{true})) = \max\left(abs\begin{bmatrix} 0.053 \\ -0.0695 \\ 0.0172 \end{bmatrix}\right) = 0.0695$$

$$||e_{app}||_\infty = \max\left(\text{abs}(e_{app})\right) = \max\left(abs\begin{bmatrix} -0.103 \\ -0.0606 \\ 0.0231 \end{bmatrix}\right) = 0.103$$

e) The error analysis shows that the Gauss-Seidel method is more accurate.

Problem 4

First, we need to find f_1 and f_2 functions, so we re-write the equations:

$$\begin{cases} \ln(x) + e^y - 8.48 = 0 \\ x^3 + x^2y^2 - 63 = 0 \end{cases}$$

$$f_1 = \ln(x) + e^y - 8.48$$

$$f_2 = x^3 + x^2y^2 - 63$$

We now are to find the $\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}^i$ for the first iteration by solving the following system:

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}^i \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}^i = -\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^i$$

$$\begin{bmatrix} \frac{1}{x} & e^y \\ x^3 + 2xy^2 & 2x^2y \end{bmatrix}^i \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}^i = -\begin{bmatrix} \ln(x) + e^y - 8.48 \\ x^3 + x^2y^2 - 63 \end{bmatrix}^i$$

1st iteration:

$$\begin{bmatrix} x \\ y \end{bmatrix}^0 = \begin{bmatrix} 3.1 \\ 2.1 \end{bmatrix}$$

$$\begin{bmatrix} 0.3225 & 8.1662 \\ 57.133 & 40.362 \end{bmatrix}^0 \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}^0 = -\begin{bmatrix} 0.8175 \\ 9.1711 \end{bmatrix}^0 \Rightarrow \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}^0 = \begin{bmatrix} -0.0924 \\ -0.0965 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}^1 = \begin{bmatrix} x \\ y \end{bmatrix}^0 + \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}^0$$

$$\begin{bmatrix} x \\ y \end{bmatrix}^1 = \begin{bmatrix} 3.1 \\ 2.1 \end{bmatrix} + \begin{bmatrix} -0.0924 \\ -0.0965 \end{bmatrix} = \begin{bmatrix} 3.0076 \\ 2.0035 \end{bmatrix}$$

2nd iteration:

$$\begin{bmatrix} x \\ y \end{bmatrix}^1 = \begin{bmatrix} 3.0076 \\ 2.0035 \end{bmatrix}$$

$$\begin{bmatrix} 0.3325 & 7.4150 \\ 51.351 & 36.2460 \end{bmatrix}^1 \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}^1 = -\begin{bmatrix} 0.0361 \\ 0.5151 \end{bmatrix}^1 \Rightarrow \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}^1 = \begin{bmatrix} -0.0068 \\ -0.0046 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}^2 = \begin{bmatrix} x \\ y \end{bmatrix}^1 + \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}^1$$

$$\begin{bmatrix} x \\ y \end{bmatrix}^2 = \begin{bmatrix} 3.0076 \\ 2.0035 \end{bmatrix} + \begin{bmatrix} -0.0068 \\ -0.0046 \end{bmatrix} = \begin{bmatrix} 3.0008 \\ 1.9989 \end{bmatrix}$$

b- true error vector and approximate relative error vector:

$[True\ error] = [Exact\ ] - [Approximation]$

$$e_{true} = \begin{bmatrix} e_x \\ e_y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 3.0008 \\ 1.9989 \end{bmatrix} = \begin{bmatrix} -0.0008 \\ 0.0011 \end{bmatrix}$$

$[Approximate\ relative\ error] = \frac{[Approximation]^2 - [Approximation]^1}{[Approximation]^2}$

$e_{app} = \begin{bmatrix} e_x \\ e_y \end{bmatrix} = \frac{\begin{bmatrix} 3.0008 \\ 1.9989 \end{bmatrix} - \begin{bmatrix} 3.0076 \\ 2.0035 \end{bmatrix}}{\begin{bmatrix} 3.0008 \\ 1.9989 \end{bmatrix}} = \begin{bmatrix} -0.0022 \\ -0.0023 \end{bmatrix}$

The 1^{st} norm is the summation of absolute elements of the error voctor:

$||e_{true}||_1 = \sum \text{abs}(e_{true}) \ = |-0.0008| + |0.0011| = 0.0019$

$||e_{app}||_1 = |-0.0022| + |-0.0023| = 0.0045$

Problem 5

***Note**: Because all the given points must be used in the interpolation, the order of the x values are also given in the problem, so we can use orders as given without any change. However, we can also reorder the data based on the distance from desired point ($x = -1.75$) and use them based on the new order. Both ways are correct.

$n + 1 = number\ of\ points$

$n + 1 = 4$ => $n = 3$ the order of the polynomial.

a- Lagrange:

$p_3(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) + L_3f(x_3)$

$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x+2.5)(x+1)(x-0.5)}{(x_0+2.5)(x_0+1)(x_0-0.5)}$ =-0.36161

$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x+3)(x+1)(x-0.5)}{(x_1+3)(x_1+1)(x_1-0.5)}$ =0.9375

$L_2(x) = \frac{(x-x_1)(x-x_0)(x-x_3)}{(x_2-x_1)(x_2-x_0)(x_2-x_3)} = \frac{(x+2.5)(x+3)(x-0.5)}{(x_2+2.5)(x_2+3)(x_2-0.5)}$ =0.46875

$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_0)}{(x_3-x_1)(x_3-x_2)(x_3-x_0)} = \frac{(x+2.5)(x+1)(x+3)}{(x_3+2.5)(x_3+1)(x_3+3)}$ =-0.04464

$p_3(-1.75) = 1.2111$

b- Newton's: (we use table method)

X	f	f$[x_i]$	f$[x_i, x_{i-1}]$	f$[x_i, x_{i-1}, x_{i-2}]$	f$[x_i, x_{i-1}, x_{i-2}, x_{i-3}]$
-3	2.2	2.2			
-2.5	1.8	1.8	$\frac{1.8-2.2}{-2.5+3} = -0.8$		
-1	0.8	0.8	$\frac{0.8-1.8}{-1+2.5} = -.6667$	$\frac{-0.6667+.8}{-1+3} = +0.0666$	
0.5	1.25	1.25	$\frac{1.25-0.8}{0.5+1} = 0.3$	$\frac{0.3+0.6667}{0.5+2.5} = 0.3222$	$\frac{0.3222-0.0666}{0.5+3} = 0.0730$

$p_3(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$

$p_3(x) = 2.2 - 0.8(x + 3) + 0.0666(x + 3)(x + 2.5) + 0.073(x + 3)(x + 2.5)(x + 1)$

$P_3(-1.75) = 1.2111$

c- Newton's method is more efficient since we do not need to start from scratch.

d- We just need to add another column to the table on part 2:

X	f	f[x_i]	f[x_i, x_{i-1}]	f[x_i, x_{i-1}, x_{i-2}]	f[$x_i, x_{i-1}, x_{i-2}, x_{i-3}$]	f[$x_i, \ldots, x_{i-4}$]
-3	2.2	2.2				
-2.5	1.8	1.8	$\frac{1.8-2.2}{-2.5+3}=-0.8$			
-1	0.8	0.8	$\frac{0.8-1.8}{-1+2.5}=-.6667$	$\frac{-0.6667+.8}{-1+3}=+0.0666$		
0.5	1.25	1.25	$\frac{1.25-0.8}{0.5+1}=0.3$	$\frac{0.3+0.6667}{0.5+2.5}=0.3222$	$\frac{0.3222-0.0666}{0.5+3}=0.0730$	
1.25	1.5	1.5	$\frac{1.5-1.25}{1.25-0.5}=0.333$	$\frac{0.333-0.3}{1.25+1}=0.0148$	$\frac{0.0148-0.3222}{1.25+2.5}=-0.0711$	$\frac{-0.0711-0.0730}{1.25+3}=-0.03391$

$$p_4(x)=a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1)+a_3(x-x_0)(x-x_1)(x-x_2)+a_4(x-x_0)(x-x_1)(x-x_2)(x-x_3)$$

$$p_4(x)=2.2-0.8(x+3)+0.0666(x+3)(x+2.5)+0.073(x+3)(x+2.5)(x+1)+0.03391(x+3)(x+2.5)(x+1)(x-0.5)$$

$$P_4(-1.75)=1.2647$$