MAST 335 / Winter 2015

Assignment 1

Due : Wednesday, January 21st, 2015 (at the beginning of the class)

Please note :
- *Each student should submit their assignments before the beginning of class on the announced due date. Late assignments will not be accepted.*
- *Solutions must be written up carefully, showing all work, for full credit.*

---

1. John deposits 15000 into a bank account that pays an effective annual interest rate of $5^o/_o$ for fifteen years, with interest credited at the end of each year. At the beginning of the second year he makes a second deposit $3^o/_o$ greater than the previous year's deposit. If a withdrawal is made during the first 10 years, John pays a penalty of $k^o/_o$ of the withdrawal amount. John withdraws 1000 at the end of each of years 9, 11 and 12. Find k such that John's bank account balance is 58027.14 at the end of fifteen years ?

   Solution. At time $t = 0$ there is a deposit of 15000. At time $t = 1$ (the beginning of the second year) there is a second deposit of $15000 \cdot (1.03) = 15450$.
   At time $t = 9$, John withdraws 1000 and hence, he pays a penalty of $1000 \cdot (0.01k)$. John withdraws 1000 at time $t = 11$ and at time $t = 12$. The balance is 58027.14 at the end of fifteen years :

$$58027.14 = 15000(1.05)^{15} + 15450(1.05)^{14} - 1000(1 + 0.01k)(1.05)^6$$

$$-1000(1.05)^4 - 1000(1.05)^3$$

   Solving the above equation for k, it follows that $k = 2.49974 \approx 2.5$.

2. If effective annual compound interest rates over an n-year period are i_1 in the first year, i_2 in the second year,..., i_n in the nth year, and the average annual compound rate of interest for the n-year period is j, then

$$(1 + j)\left(\frac{1}{1 + i_1} + \frac{1}{1 + i_2} + ... + \frac{1}{1 + i_n}\right) \geq n.$$

Solution. If j is the average annual compound rate of interest for the n-year period, then

$$(1+j)^n = (1+i_1)(1+i_2)...(1+i_n) \quad \Rightarrow \quad 1+j = \left[(1+i_1)(1+i_2)...(1+i_n)\right]^{\frac{1}{n}}.$$

Using the inequality between harmonic mean and geometric mean of the positive numbers $(1+i_1)$, $(1+i_2)$,..., and $(1+i_n)$ leads to

$$\frac{n}{\frac{1}{1+i_1}+\frac{1}{1+i_2}+...+\frac{1}{1+i_n}} \leq \left[(1+i_1)(1+i_2)...(1+i_n)\right]^{\frac{1}{n}} = 1+j.$$

Therefore,

$$(1+j)\left(\frac{1}{1+i_1}+\frac{1}{1+i_2}+...+\frac{1}{1+i_n}\right) \geq n.$$

3. Suppose that an amount of $A(0)$ accumulates to $A(1) = 105$ at time $t = 1$, to $A(2) = 110.25$ at time $t = 2$, and to $A(10) = 162.8894627$ at time $t = 10$.
(a) Find the effective rate of discount during the second year.
Solution. The effective rate of discount during the second year is

$$d_2 = \frac{A(2)-A(1)}{A(2)} = \frac{110.25-105}{110.25} = 0.047619.$$

(b) Is the interest model a simple interest model or a compound interest model ? Justify the answer.
Solution. Assume simple interest model. If i is an annual simple interest rate, then

$$A(1) = A(0)(1+i) = 105;$$

$$A(2) = A(0)(1+2i) = 110.25;$$

$$A(10) = A(0)(1+10i) = 162.8894627.$$

Solving the first two equations in $A(0)$ and i results in $\frac{1+2i}{1+i} = \frac{110.25}{105} = 1.05$, and therefore, $i = 0.0526$ and $A(0) = \frac{105}{1+i} = 99.75$. By using these values, it results that $A(10) = A(0)(1+10i) = 152.2185$, which contradicts the third equation.
Assume compound interest model. If i is an annual compound interest rate, then

$$A(1) = A(0)(1+i) = 105;$$

$$A(2) = A(0)(1+i)^2 = 110.25;$$

$$A(10) = A(0)(1+i)^{10} = 162.8894627.$$

Solving the first two equations in i and $A(0)$ results in $1+i = \frac{110.25}{105} = 1.05$ ($i = 0.05$) and $A(0) = \frac{105}{1+i} = \frac{105}{1.05} = 100$, and hence, $A(10) = A(0)(1+i)^{10} = 100(1.05)^{10} = 162.8894627$, which satisfies the third equation.
So, the model is a compound interest model.

4. At a certain rate of simple interest 1000\$ will accumulate to 1110\$ after a certain period of time. Find the accumulated value of 500\$ at a rate of simple interest three fourths as great over twice as long a period of time.

   Solution. $a(t) = 1 + it$ for simple interest, $1000a(t) = 1110$. Therefore, $1000(1 + it) = 1110$ and hence $it = 0.11$.
   Now, $500(1 + \frac{3}{4}i(2t)) = 500(1 + \frac{3}{2}it) = 500(1 + 3/2(0.11)) = 582.50$.

5. (a) At an effective annual interest rate of 6%, calculate the number of years it will take for an investment of 2,000 to accumulate to 5,000. (b) Repeat part (a) using simple interest only for fractions of a year (answer in days). (c) Repeat part (a) using an effective monthly interest rate of 0.5%, is it faster than in (a) ? (d) Suppose that an investment of $2,000$ accumulates to 5,000 in exactly 10 years at an annual effective rate of interest i. Calculate i. (e) Repeat part (d) using a monthly effective rate of interest j. Calculate j. <u>Solution :</u>

   a) $5000 = 2000(1.06)^t \rightarrow t = \frac{ln(2.5)}{ln(1.06)} = 15.725$ years. So 15 years and 265 days.
   b) $2000(1.06)^{15} = 4793.12$ and he needs the number of days using simple interest for the portion of the 16th year. Hence, $4793.12(1 + t * 0.06) \rightarrow t = 0.7193644$ which is 262 days.
   c) $j = 0.005$, so $2000(1 + j)^t = 5000 \rightarrow t = \frac{ln(5/2)}{ln(1.005)} = 183.7159$ months. This means 15 years, 3 months and 22 days. So it is faster than in (a).
   d) $2.5 = (1 + i)^{10} \rightarrow i = 2.5^{0.1} - 1 = 9.596\%$
   e) $2.5 = (1+j)^{120} \rightarrow j = (2.5)^{1/120} - 1 = 0.007664983$. So the effective monthly interest rate $j = .766\%$.