## Solution to Assignment 5 (4001)

1. (Part b in Problem 1 on page 338) Find the only possible solutions to the following variable final time

$$\begin{cases} \max\limits_{u,\ T} \int_0^T \left[-9 - \frac{1}{4}u^2(t)\right] dt \\ \dot{x}(t) = u(t) \in U(t) = R^1 \\ x(0) = 0,\ x(T) = 16. \end{cases}$$

Step 1: For any given $T > 0$, define the Hamiltonian

$$H(t, x, u, \lambda) = -9 - \frac{1}{4}u^2 + \lambda u.$$

Step 2: Solve for $u^*$ from

$$\max_{u \in R^1} H(t, x, u, \lambda) = -9 - \frac{1}{4}u^2 + \lambda u.$$

From

$$H'_u(t, x, u, \lambda) = -\frac{u}{2} + \lambda = 0,$$

we have

$$u = 2\lambda.$$

Since

$$H''_{u^2}(t, x, u, \lambda) = -\frac{1}{2} < 0,$$

$H$ is concave in $u$. Thus, $u = 2\lambda$ solves

$$\max_{u \in R^1} H(t, x, u, \lambda) = -9 - \frac{1}{4}u^2 + \lambda u.$$

Step 3: Derive

$$\dot{\lambda} = -H'_x.$$

Since $H'_x = 0$,

$$\dot{\lambda} = 0.$$

Step 4: Solve for $x^*, u^*, \lambda^*$.

$$\begin{aligned} u &= 2\lambda \\ \dot{\lambda} &= 0 \\ \dot{x} &= u,\ x(0) = 0,\ x(T) = 16. \end{aligned}$$

From $\dot{\lambda} = 0$,

$$\begin{aligned} \lambda^* &= C \text{ (constant to be decided)} \\ u^* &= 2C. \end{aligned}$$

From $\dot{x} = u = 2C$, $x(0) = 0$, $x(T) = 16$,

$$\begin{aligned} x(t) &= 2Ct + A \\ x(0) &= 2C \times 0 + A = 0 \Longrightarrow A = 0 \\ x(T) &= 2CT = 16 \Longrightarrow C = \frac{8}{T} \\ x(t) &= 16\frac{t}{T} \end{aligned}$$

That is,

$$\begin{aligned} x^* &= 16\frac{t}{T} \\ u^* &= \frac{16}{T} \\ \lambda^* &= \frac{8}{T}. \end{aligned}$$

Step 5: Decide if the possible solution is the solution.

Since

$$H(t, x, u, \lambda) = -9 - \frac{1}{4}u^2 + \lambda^* u$$

is concave in $(x, u)$, thus the possible solution it the optimal solution for given terminal time $T$.

Step 6. Decide $T^*$ from

$$H(T, x^*(T), u^*(T), \lambda^*(T)) = 0.$$

Since

$$\begin{aligned} x^*(T) &= 16\frac{T}{T} = 16 \\ u^*(T) &= \frac{16}{T} \\ \lambda^*(T) &= \frac{8}{T}, \end{aligned}$$

$$\begin{aligned} H(T, x^*(T), u^*(T), \lambda^*(T)) &= -9 - \frac{1}{4}(u^*(T))^2 + \lambda^*(T)\, u^*(T) \\ &= -9 - \frac{1}{4}\left(\frac{16}{T}\right)^2 + \frac{8}{T} \times \frac{16}{T} \\ &= -9 - \frac{4 \times 16}{T^2} + \frac{8 \times 16}{T^2} = \frac{64 - 9T^2}{T^2} = 0. \end{aligned}$$

That is

$$T^* = \sqrt{\frac{64}{9}} = \frac{8}{3}.$$

Finally, the possible solution is

$$T^* = \frac{8}{3}$$

and

$$\begin{aligned} x^* &= 16\frac{t}{\frac{8}{3}} = 6t, \ 0 \le t \le \frac{8}{3} \\ u^* &= \frac{16}{T^*} = 6 \\ \lambda^* &= \frac{8}{T^*} = 6. \end{aligned}$$

1. (Part a in Problem 4 on page 347)

$$\begin{cases} \max_u \left\{ \int_0^1 [x(t) - u(t)]\, dt + \frac{1}{2} x(1) \right\} \\ \dot{x}(t) = u(t), x(0) = \frac{1}{2}, \ x(1) \text{ free} \\ u(t) \in U(t) = [0, \ 1]. \end{cases}$$

Step 1: Define the Hamiltonian

$$H(t, x, u, \lambda) = x - u + \lambda u.$$

Step 2: Solve

$$\max_{u \in [0,1]} H(t, x, u, \lambda) = x - u + \lambda u.$$

From

$$H(t, x, u, \lambda) = x - u + \lambda u = x + u(\lambda - 1),$$

we have

$$u = \begin{cases} 1 & \text{if } \lambda > 1 \\ 0 & \text{if } \lambda < 1 \end{cases}.$$

Step 3: Derive

$$\dot{\lambda} = -H'_x.$$

Since $H'_x = 1$,

$$\dot{\lambda} = -1.$$

Since

$$\begin{aligned} S(x(1)) &= \frac{1}{2} x(1), \ S'(x(1)) = \frac{1}{2} \\ \lambda(1) &= S'(x(1)) = \frac{1}{2}. \end{aligned}$$

Step 4: Solve for $x^*, u^*, \lambda^*$.

$$\begin{aligned} u &= \begin{cases} 1 & \text{if } \lambda > 1 \\ 0 & \text{if } \lambda < 1 \end{cases} \\ \dot{\lambda} &= -1,\ \lambda(1) = \frac{1}{2} \\ \dot{x} &= u,\ x(0) = \frac{1}{2}. \end{aligned}$$

From $\dot{\lambda} = -1,\ \lambda(1) = \frac{1}{2}$,

$$\begin{aligned} \lambda^* &= -t + A \Longrightarrow \frac{1}{2} = -1 + A \Longrightarrow A = \frac{3}{2} \\ \lambda^* &= \frac{3}{2} - t \end{aligned}$$

and

$$u^* = \begin{cases} 1 & \text{if } \lambda > 1 \\ 0 & \text{if } \lambda < 1 \end{cases} = \begin{cases} 1 & \text{if } \frac{3}{2} - t > 1 \\ 0 & \text{if } \frac{3}{2} - t < 1 \end{cases} = \begin{cases} 1 & \text{if } t < \frac{1}{2} \\ 0 & \text{if } t > \frac{1}{2} \end{cases}$$

From $\dot{x} = u,\ x(0) = \frac{1}{2}$,

$$x^* = \begin{cases} t + A & \text{if } t < \frac{1}{2} \\ B & \text{if } t > \frac{1}{2} \end{cases}$$

From $x(0) = \frac{1}{2}$, $A = \frac{1}{2}$. Since $x(t)$ is continuous at $\frac{1}{2}$,

$$\frac{1}{2} + \frac{1}{2} = B = 1.$$

That is,

$$x^* = \begin{cases} t + \frac{1}{2} & \text{if } t < \frac{1}{2} \\ 1 & \text{if } t > \frac{1}{2} \end{cases}.$$

Step 5. Verify if $(x^*, u^*)$ is the solution.

From

$$H(t, x, u^*, \lambda^*) = x + u^*(\lambda^* - 1) = \begin{cases} x + \left(\frac{1}{2} - t\right) & \text{if } t < \frac{1}{2} \\ x & \text{if } t > \frac{1}{2}, \end{cases}$$

$H(t, x, u^*, \lambda^*)$ is concave in $x$. Thus, the only possible solution is the solution.

2. (Problem 4 on page 353) Solve the problem

$$\begin{cases} \max_u \int_{-1}^{+\infty} [x(t) - u(t)]\, e^{-t} dt \\ \dot{x}(t) = u(t)\, e^{-t}, x(-1) = 0,\ x(+\infty) \text{ free} \\ u(t) \in U(t) = [0,\ 1]. \end{cases}$$

Step 1: Define the current value Hamiltonian

$$H^c(t, x, u, \lambda) = x - u + \lambda^c u e^{-t}.$$

Step 2: Solve

$$\max_{u\in[0,1]} H^c(t,x,u,\lambda) = x - u + \lambda^c u e^{-t}.$$

From

$$H^c(t,x,u,\lambda) = x - u + \lambda^c u e^{-t} = x + u\left(\lambda^c e^{-t} - 1\right),$$

we have

$$u = \begin{cases} 1 & \text{if } \lambda^c e^{-t} > 1 \\ 0 & \text{if } \lambda^c e^{-t} < 1 \end{cases} = \begin{cases} 1 & \text{if } \lambda^c > e^t \\ 0 & \text{if } \lambda^c < e^t \end{cases}.$$

Step 3: Derive

$$\dot{\lambda}^c - \lambda^c = -H_x^{c\prime}.$$

Since $H_x^{c\prime} = 1$,

$$\dot{\lambda}^c - \lambda^c = -1 \text{ and } \lim_{t\to+\infty} e^{-t}\lambda^c(t) = 0.$$

Step 4: Solve for $x^*, u^*, \lambda^*$.

$$u = \begin{cases} 1 & \text{if } \lambda^c > e^t \\ 0 & \text{if } \lambda^c < e^t \end{cases}$$

$$\begin{aligned} \dot{\lambda}^c - \lambda^c &= -1 \text{ and } \lim_{t\to+\infty} e^{-t}\lambda^c(t) = 0 \\ \dot{x}(t) &= u(t)e^{-t},\ x(-1) = 0,\ x(+\infty) \text{ free} \end{aligned}$$

From $\dot{\lambda}^c - \lambda^c = -1$,

$$\lambda^c = Ae^t + 1.$$

From $\lim_{t\to+\infty} e^{-t}\lambda^c(t) = 0$,

$$\lim_{t\to+\infty} e^{-t}\lambda^c(t)e^{-t}\lambda^c = \lim_{t\to+\infty} e^{-t}\left[Ae^t + 1\right] = A + \lim_{t\to+\infty} e^{-t} = 0 \Longrightarrow A = 0.$$

That is,

$$\lambda^{*c} = 1.$$

Thus,

$$u^* = \begin{cases} 1 & \text{if } 1 > e^t \\ 0 & \text{if } 1 < e^t \end{cases} = \begin{cases} 1 & \text{if } -1 \le t \le 0 \\ 0 & \text{if } t > 0 \end{cases}.$$

From $\dot{x}(t) = u(t)e^{-t}, x(-1) = 0$,

$$\begin{aligned} \dot{x}(t) &= u(t)e^{-t} = \begin{cases} e^{-t} & \text{if } -1 \le t \le 0 \\ 0 & \text{if } t > 0 \end{cases} \\ x^*(t) &= \begin{cases} -e^{-t} + e & \text{if } -1 \le t \le 0 \\ e - 1 & \text{if } t > 0. \end{cases} \end{aligned}$$

Step 5. Verify if $(x^*, u^*, \lambda^{*c})$ is the solution.

From

$$H^c(t,x,u^*,\lambda^{c*}) = x + u\left(\lambda^c e^{-t} - 1\right) = \begin{cases} x + e^{-t} - 1 & \text{if } -1 \le t \le 0 \\ x & \text{if } t > 0. \end{cases}$$

$H^c(t,x,u^*,\lambda^{c*})$ is concave in $x$. Thus, the only possible solution is the solution.