Solutions.

Final Exam
Mat2322 C - April 2008

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INSTRUCTIONS.
1. Exam period: 3 hours.
2. No books, no notes and **no calculators** are allowed.
3. There are 10 questions to be answered.
4. You have to justify that your answer is correct.

GOOD LUCK!

Question 1. Consider the function

$$f(x,y) = xy + \frac{8}{x^2} + \frac{8}{y^2}.$$

Find all the critical points and classify them as saddle, maximum or minimum. Notice that the domain of f does <u>not</u> include $x = y = 0$.

$$f(x,y) = xy + 8x^{-2} + 8y^{-2}$$

$f_x = y - 16x^{-3}$, $f_y = x - 16y^{-3}$. If $\nabla f = \vec{0}$ then

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \rightarrow \begin{cases} y = \frac{16}{x^3} \quad (1) \rightarrow x^3 y = 16 \quad (3) \\ x = \frac{16}{y^3} \quad (2) \end{cases}$$

Plug in (3) into (2) $\quad \left(\frac{16}{y^3}\right)^3 y = 16 \rightarrow \frac{16^2}{y^8} = 1 \rightarrow y^8 = 2^8$

$\Rightarrow y = \pm 2$. if $y = +2 \rightarrow x = 2$ & $y = -2 \rightarrow x = -2$.

The critical points are $(-2,-2)$, $(2,2)$.

$f_{xx} = 48x^{-4} \quad f_{yy} = 48y^{-4} \quad f_{xy} = 1$

$D = 48^2 x^{-4} y^{-4} - 1$ at $(-2,-2)$ or $(2,2)$ $D > 0$ Moreover,

$$f_{xx}(-2,-2) = f_{xx}(2,2) = \frac{48}{(\pm 2)^4} = 3 > 0$$

$\therefore (2,2), (-2,-2)$ are local minimums.

Question 2. Use Lagrange Multiplier method to find the local maximum and minimum values of the function

$$f(x, y, z) = x - 2y + 5z$$

subject to the constraint $g(x, y) = x^2 + y^2 + z^2 = 30$. [Hint: There are only one max and one min.]

Solve $\begin{cases} \nabla f = \lambda \nabla g \\ g = 30 \end{cases}$ $\qquad \nabla f = \vec{i} - 2\vec{j} + 5\vec{k}$

$\nabla g = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

the first eq. is $1 = 2\lambda x$, $-2 = 2\lambda y$, $5 = 2\lambda z$

$\therefore$ $x = \frac{1}{2\lambda}$ $y = \frac{-1}{\lambda}$, $z = \frac{5}{2\lambda}$ plug in these in $g = 30$.

$$30 = \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{-1}{\lambda}\right)^2 + \left(\frac{5}{2\lambda}\right)^2 = \frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{25}{4\lambda^2} = \left(\frac{1}{4} + \frac{4}{4} + \frac{25}{4}\right) \frac{1}{\lambda^2}$$

$4\lambda^2 = 1 \longrightarrow \lambda = \pm\frac{1}{2}$.

if $\lambda = \frac{1}{2} \longrightarrow x = 1, y = -2, z = 5$

if $\lambda = \frac{-1}{2} \longrightarrow x = -1, y = 2, z = -5$

$f(1, -2, 5) = 1 - 2(-2) + (5)(5) = 30 \longleftarrow$ local max

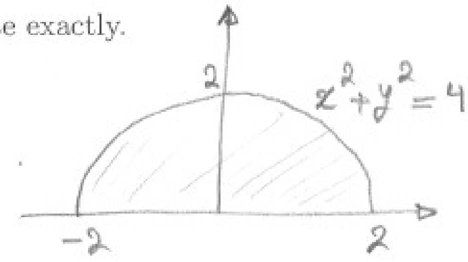
$f(-1, 2, -5) = -1 - 2(2) + 5(-5) = -30 \longleftarrow$ local min

Question 3. Convert the integral

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} e^{-(x^2+y^2)}\,dy\,dx,$$

to polar coordinates and evaluate exactly.

The region is



$$\int_{\theta=0}^{\pi}\int_{0}^{2} e^{-r^2}\,r\,dr\,d\theta = \int_{0}^{\pi} \left.-\frac{1}{2}e^{-r^2}\right]_0^2 d\theta =$$

$$\frac{\pi}{2}\left(1-e^{-4}\right).$$

Question 4. Find the mass of the solid which is bounded between the two spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 9$ with density function

$$f(x, y, z) = \frac{1}{x^2 + y^2 + z^2}.$$

Using spherical coordinates:

$$0 \leqslant \phi \leqslant \pi$$

$$0 \leqslant \theta \leqslant 2\pi$$

radius of inner sphere $\rightarrow 1 \leqslant \rho \leqslant 3 \leftarrow$ radius of outer sphere.

$$\text{Mass} = \int_W \frac{1}{x^2+y^2+z^2}\, dV = \int_0^{2\pi}\int_0^{\pi}\int_1^3 \frac{1}{\rho^2}(\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} (3-1) \sin\phi\, d\phi\, d\theta =$$

$$2\int_0^{2\pi} \left[-\cos\phi\right]_0^{\pi} d\theta = 2(+1+1)\int_0^{2\pi} d\theta = 8\pi.$$

Question 5. Calculate $\int_C \vec{F} \cdot d\vec{r}$ when

$$\vec{F} = (y+z)\vec{i} + (x+z)\vec{j} + \vec{k},$$

and C is the line from the origin to the point $(3,3,3)$.

A parametrization of the line is

$$\vec{r}(t) = \underbrace{(0,0,0)}_{\text{initial point}} + \underbrace{(3,3,3)}_{\substack{(3,3,3)-(0,0,0) \\ \text{parallel to the line}}} t = 3t\vec{i} + 3t\vec{j} + 3t\vec{k}$$

where $0 \leq t \leq 1$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t=0}^{1} \vec{F}(3t,3t,3t) \cdot \vec{r}\,'(t)\, dt$$

$$[6t\vec{i} + 6t\vec{j} + \vec{k}] \cdot [3\vec{i} + 3\vec{j} + 3\vec{k}] = 18t + 18t + 3 = 36t + 3$$

So

$$\int_C \vec{F} \cdot d\vec{r} = 3\int_0^1 (12t+1)\, dt = 3\left(6t^2 + t\right)\Big|_0^1 = 3(6+1) = 21.$$

Question 6. Given that

$$\vec{G} = 3x\,\vec{i} + y^2\,\vec{j},$$

compute $\int_C \vec{G} \cdot d\vec{r}$, where C is part of a hyperbola beginning at the point $(1,4)$ and ending at the point $(0,5)$. (Messy calculation can be avoided.)

Notice that $\operatorname{curl} \vec{G} = \frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(3x) = 0 - 0 = 0$ &

$\vec{G}$ is defined everywhere

$\therefore$ $\vec{G}$ is a gradient field. f exists s.t. $\vec{G} = \nabla f$.

$$\begin{cases} f_x = 3x \longrightarrow f = \frac{3}{2}x^2 + C(y) \longrightarrow f_y = C'(y) \\ f_y = y^2 \end{cases}$$

So from the 2nd. eq. $C'(y) = y^2 \longrightarrow C(y) = \frac{1}{3}y^3$

$$\Longrightarrow f = \frac{3}{2}x^2 + \frac{1}{3}y^3$$

$$\int_C \vec{G} \cdot d\vec{r} = f(0,5) - f(1,4) = \frac{5^3}{3} - \left(\frac{3}{2} + \frac{4^3}{3}\right) = \frac{125}{3} - \frac{3}{2} - \frac{64}{3}$$

$$= \frac{61}{3} - \frac{3}{2} = \frac{122-9}{6} = \frac{113}{6}$$

Question 7. A moving particle has position vector

$$\vec{r}(t) = \left(\frac{t^2}{2} - \frac{t^4}{4}\right)\vec{i} + \frac{2t^3}{3}\vec{j} + \vec{k}, \text{ for } -1 \le t \le 1.$$

(a) Find the velocity vector of the particle at time t.
(b) What is the distance traveled by the particle during the interval $-1 \le t \le 1$? Use the fact that $\|\vec{v}(t)\| = \sqrt{t^2(1+t^2)^2}$.

a) $v(t) = \vec{r}\,'(t) = (t - t^3)\vec{i} + 2t^2\vec{j} + 0\vec{k}$

b) $\|\vec{v}(t)\| = \begin{cases} t(1+t^2) & t \geq 0 \\ -t(1+t^2) & t < 0 \end{cases}$

distance traveled $= \displaystyle\int_{t=-1}^{t=1} \|\vec{v}(t)\|\,dt = \int_{-1}^{0} [-t(1+t^2)]\,dt + \int_0^1 t(1+t^2)\,dt$

$= \left(-\frac{t^2}{2} - \frac{t^4}{4}\right)\Big]_{-1}^{0} + \left(\frac{t^2}{2} + \frac{t^4}{4}\right)\Big]_0^1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = \frac{2+1+2+1}{4}$

$= \frac{6}{4} = \frac{3}{2}$

Question 8. Compute the flux of the vector field

$$\vec{F} = (x - 2y)\,\vec{i} + (4y - 2z)\,\vec{j} + 3x\,\vec{k},$$

through the surface S, where S is the part of the plane $z = x + 2y$ above the rectangle $0 \leq x \leq 2$, $0 \leq y \leq 3$, oriented upward.

$z = f(x,y) = x + 2y$

$$\int_S \vec{F} \cdot d\vec{A} = \int_{xy\text{-region}} \vec{F} \cdot (-f_x \vec{i} - f_y \vec{j} + \vec{k})\, dx\, dy =$$

$$\int_{x=0}^{2} \int_{y=0}^{3} [(x-2y)\vec{i} + (4y-2z)\vec{j} + 3x\vec{k}] \cdot (-\vec{i} - 2\vec{j} + \vec{k})\, dy\, dx =$$

$$\int_{x=0}^{2} \int_{y=0}^{3} \underbrace{(2y - x - 8y + 4z + 3x)}_{\underbrace{-6y + 2x + 4(x+2y)}_{2y + 6x}}\, dy\, dx = \int_0^2 \underbrace{\int_0^3 (2y + 6x)\, dy}_{y^2 + 6xy \big]_0^3 = 9 + 18x}\, dx$$

$$= \int_0^2 9 + 18x\, dx = 9x + 9x^2 \Big]_0^2 = 18 + 36 = 54.$$

Question 9. Use the Divergence Theorem to evaluate the flux integral

$$\int_S \underbrace{[x^3 \vec{i} + (y^3 + \cos z)\vec{j} + (z^3 + x^2 + y^2)\vec{k}]}_{\vec{F}} . d\vec{A},$$

where S is the surface of the cylinder $x^2 + y^2 = 1$ bounded by the planes $z = -1$ and $z = 1$ (including the ends of the cylinder). It may be helpful to use cylindrical coordinates somewhere.

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3 + \cos z) + \frac{\partial}{\partial z}(z^3 + x^2 + y^2)$$

$$= 3x^2 + 3y^2 + 3z^2$$

Cyl. coord.

$$\int_S \vec{F}.d\vec{A} = \int_W \operatorname{div}\vec{F}\, dV = \int (3r^2 + 3z^2)(r\,dr\,d\theta\,dz)$$

$$= \int_{z=-1}^{1} \int_0^1 \int_{\theta=0}^{2\pi} (3r^3 + 3z^2 r)\, d\theta\, dr\, dz =$$

$$6\pi \int_{-1}^1 \underbrace{\int_0^1 (r^3 + z^2 r)\, dr}_{\frac{1}{4}r^4 + \frac{1}{2}z^2r^2\Big]_0^1 \,=\, \frac{1}{4} + \frac{1}{2}z^2} dz = 6\pi \int_{-1}^1 \frac{1}{4} + \frac{1}{2}z^2\, dz$$

$$= 6\pi\left(\frac{1}{4}z + \frac{1}{6}z^3\right)\Big|_{-1}^1 =$$

$$6\pi\left(\frac{1}{4} + \frac{1}{6} + \frac{1}{4} + \frac{1}{6}\right) =$$

$$6\pi\left(\frac{1}{2} + \frac{1}{3}\right) = 6\pi\,\frac{3+2}{6}$$

$$= 5\pi$$

Question 10. Let $\vec{F} = xz\,\vec{i} + (y-x)\,\vec{j} + x\,\vec{k}$.

(a) Compute $\operatorname{curl}\vec{F}(0,0,0)$.

(b) Use the Stokes' Theorem to find the circulation of $\vec{F}$ around the circle, $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + 0\,\vec{k}$, for $0 \le t \le 2\pi$.

a) $$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & y-x & x \end{vmatrix} =$$

$$(0-0)\vec{i} - (1-x)\vec{j} + (-1-0)\vec{k} =$$

$$(x-1)\vec{j} - \vec{k}$$

b) Let S be the disk in the xy-plane enclosed by C

The unit orientation vector of S is $\vec{n}$

$$\therefore\ d\vec{A} = \vec{k}\,dA$$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_S \operatorname{curl}\vec{F}\cdot d\vec{A} = \int_S \left[(x-1)\vec{j} - \vec{k}\right]\cdot\vec{k}\,dA =$$

$$\int_S (-1)\,dA = (-1)\,(\text{area of } S) = -(1\times\pi^{\cancel{2}}) = -\pi^{\cancel{2}}.$$