MAT1320 – Notes — By Eric Hua

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Chapter 1 – Functions and Models

1.1 Four Ways to Represent A Function

Function: *A function $y = f(x)$ from a set D to a set Y is a rule that assigns a unique element $f(x) \in Y$ to each element $x \in D$. (x is called independent variable, y is called dependent variable).*

- Domain of the function $y = f(x)$: $D =$ The set of all values of the independent variable x for which the function is defined.
- Range of the function: $R =$ The set of all values taking on by the dependent variable y.

Four ways to represent functions:

- Verbally: by a description in words;
- numerically: by a table of values;
- visually: by a graph $\{(x, f(x))|x \in D\}$;
- algebraically: by an explicit formula.

Example: $f(x) = \frac{x^2}{x^2-3x+2}$ is a function, $D = \{x : x \neq 1, 2\}$.
Example: $f(x) = \pm x^2$ is not a function.
Some special functions:

- Linear function: $y = f(x) = mx + b$.
- Increasing function $f(x)$: $f(x)$ increases as x increases.
- Decreasing function $f(x)$: $f(x)$ decreases as x increases.
- Piecewise defined functions: $f(x) = \begin{cases} 2x, & x \leq 0; \\ 3x, & x > 0. \end{cases}$
- Odd functions: $f(-x) = -f(x)$; even functions: $f(-x) = f(x)$.

1.2 Mathematical Models: A Catalog of Essential functions

- Algebraic function: functions using algebraic operations (+, -, x,division and taking roots) starting with polynomials. E.g.,
  - Power function: $f(x) = kx^p$, where $k \neq 0$ and p are constants.
    Domain of $f(x) = x^{1/2}$: $\{x \geq 0\}$;
    Domain of $f(x) = x^{-1/2}$: $\{x > 0\}$;
    Domain of $f(x) = x^{-1/3}$: $\{x \neq 0\}$.
  - Polynomials $P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$, where n is a positive integer (which is called the degree of $P(x)$).
  - Rational function: $f(x) = \frac{p(x)}{q(x)}$. Domain $q(x) \neq 0$.
  - $\sqrt{1-x^2}$: $D = \{x : -1 \leq x \leq 1\}$.
  - Absolute value:
    $$|x| = \begin{cases} -x & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}$$
- Transcendental functions (non-algebraic): trig functions, exponential functions, logarithmic functions,...

1.3 New Functions from Old Functions

New Functions:

- Vertical stretch by a factor of c: $f(x) \to cf(x)$, $c > 1$;
- Vertical compress by a factor of $1/c$: $f(x) \to cf(x)$, $0 < c < 1$;
- Horizontal stretch by a factor of $1/c$: $f(x) \to f(cx)$, $0 < c < 1$;
- Horizontal compress by a factor of c: $f(x) \to f(cx)$, $c > 1$;
- Reflection about x-axis: $f(x) \to -f(x)$;
- Reflection about y-axis: $f(x) \to f(-x)$;
- Vertical shift up (or down) by k: $f(x) \to f(x) + k$, $k > 0$ (or $k < 0$);
- Horizontal shift to the right (or left) by h: $f(x) \to f(x - h)$, $h > 0$ (or $h < 0$);
- Combinations of functions, e.g., $f(x)g(x)$;
- Composite function $f(g(x))$ or $f \circ g(x)$. The domain of $f \circ g(x)$ is the set of all $x \in D(g)$ such that $g(x) \in D(f)$.

Example 1 *Let $f(x) = \sqrt{2x-4}$. Find the new function after shifting downward by 2 units, then shifting left by 3 units, then compressing vertically by 4 units.*

Example 2 *Let $f(x) = \sqrt{2x-4}$, $g(x) = \sqrt{3-2x}$.*

Then $f(g(x)) = \sqrt{2\sqrt{3-2x}-4}$ with $D(f \circ g) = \{x \leq -0.5\}$ (the solution of $3 - 2x \geq 0$ and $2\sqrt{3-2x} - 4 \geq 0$). $g(f(x)) = \sqrt{3-2\sqrt{2x-4}}$, $D(g \circ f) = \{2 \leq x \leq 3.125\}$ (the solution of $2x - 4 \geq 0$ and $3 - 2\sqrt{2x-4} \geq 0$).

1.5 Exponential functions

We say that $f(x) = a^x$ is an exponential function with base a.

- Domain: $x \in \mathbb{R}$; Range: $y > 0$.
- Exponential growth: $a > 1$; Exponential decay: $0 < a < 1$.
- Natural exponential function is defined as: $y = f(x) = e^x$, where $e \doteq 2.71828...$.

Laws of exponents:

$$a^{x+y} = a^x a^y, \quad a^{x-y} = a^x/a^y, \quad (a^x)^y = a^{xy}, \quad a^x b^x = (ab)^x.$$

Example 3 *Solve for x:* $3^{2x-3} = 9^{1-3x}$, $2^{2x+1} - 9(2^x) + 4 = 0$.

Applications:

1. Population growth/decay: Let $P(t)$ be the population after t years,
   - Half-life (exponential decay): The time required for the quantity to be reduced to half. Let H be the half-life, then
   $$P(t+H) = \frac{1}{2}P(t) \Rightarrow P(t) = P_0(\frac{1}{2})^{t/H}.$$
   - Doubling-time (exponential growth): The time required for the quantity to be doubled. Let D be the doubling time, then
   $$P(t+D) = 2P(t) \Rightarrow P(t) = P_0(2)^{t/D}.$$
2. Compounded interest: Let $P(t)$ represent the amount of money after t years, P_0 represent the principle or the amount of money you start with (Present value), r represent the annual interest rate. Then
   - Compounded n times per year:
   $$P(t) = P_0\left(1 + \frac{r}{n}\right)^{nt}.$$

- Compounded continuously:

$$P(t) = P_0 e^{rt}.$$

Example 4 *Suppose an amount \$1000 is deposit in a bank account that offers annual interest rate 5% compounded continuously. Find the balance after t years.*

Example 5 *A bacterial culture starts with 500 bacteria and doubles in size every hour.*
a) How many are there after t hours?
b) How many are there after 10 minutes?

Example 6 *Suppose \$12000 is put into an account that pays 2.4% annually. How much will be in the account after 10 years?*
a) compounded continuously.
b) compounded quarterly. What is the effective rate?

Example 7 *Sketch the graph of* $y = 2^x + 5$.

Example 8 *Sketch the graph of* $y = 2^{-x} + 5$.

1.6 Inverse functions and Logarithms

Inverse function: One-to-one function: $y = f(x)$ is 1-1 $\Leftrightarrow$ for each $y \in R$, there is only one $x \in D$. Horizontal line test can be used to check this.
Example 9 $f(x) = x^2$ *is not 1-1;* $g(x) = x^2$, $x > 0$ *is 1-1.*

Inverse function: $y = f(x) \to x = f^{-1}(y)$. We write it as $y = f^{-1}(x)$.

- The graph of f^{-1} and the graph of f are symmetric about the line $y = x$.
- Cancelation: $f(f^{-1}(y)) = y$.
- $f^{-1}(f(x)) = x$
- $D(f) = R(f^{-1})$, $R(f) = D(f^{-1})$.

Example 10 *let* $f(x) = \frac{3x+2}{5x-4}$, *find the inverse* $f^{-1}(x)$.

Strategy:
1) Write $y = \frac{3x+2}{5x-4}$;
2) Switch x and y: $x = \frac{3y+2}{5y-4}$;
3) Isolate y: $y = \frac{4x+2}{5x-3}$;
4) Answer: $y = f^{-1}(x) = \frac{4x+2}{5x-3}$.

$$y = a^x \xrightarrow{inverse function} y = \log_a x,$$
$$y = e^x \xrightarrow{inverse function} y = \log_e x = \ln x,$$
$$y = 10^x \xrightarrow{inverse function} y = \log_{10} x = \log x.$$

Definition: $y = \log_a x$ is called logarithmic function with the base a. Domain $= \{x > 0\}$.
Properties: Let $B, C > 0$. Then

1. $\log_a(BC) = \log_a B + \log_a C$,
2. $\log_a(\frac{B}{C}) = \log_a B - \log_a C$,
3. $\log_a(B^n) = n \log_a B$,
4. $\log_a(a^x) = x$, $\log_a a = 1$,
5. $a^{\log_a B} = B$,
6. $\log_a 1 = 0$.
7. Change of base: $\log_a b = \frac{\log_c b}{\log_c a}$.

Proof. Let $x = \log_a b$. Then $a^x = b \Rightarrow \log_c a^x = \log_c b \Rightarrow x \log_c a = \log_c b$.

Example 11 *Convert a^x to base e.*

$$a^x = e^{x \ln a}.$$

Example 12 *Simplify* $\log_3 18 - \log_3 2$.

Example 13 *Solve for x:*

$$3^{2x-1} = 4, \quad \ln[\ln(2x+1)] = 1, \quad \log_3 x + \log_3(x-8) = 2.$$

Example 14 *Sketch* $y = \ln(x+1) - 2$.

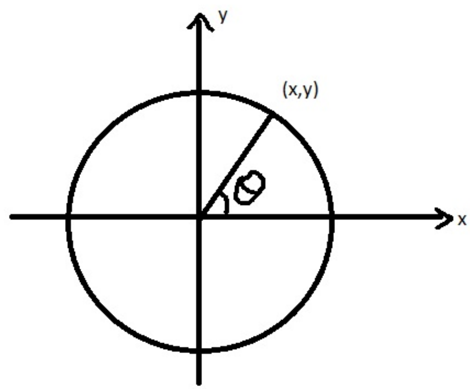
Example 15 *Predict the population in 2010, if*

Year	*Population*
2000	*10*
2003	*10.5*

Appendix D: Trigonometric functions

Radian $\leftrightarrows$ Degree: t degree $= \frac{t}{180}\pi$.

For any point (x, y), let $r = \sqrt{x^2 + y^2}$.



$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad \tan\theta = \frac{\sin\theta}{\cos\theta},$$
$$\sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta}.$$

Basic relations:

$\sin(\theta + \frac{\pi}{2}) = +\cos\theta$	$\sin(\theta + \pi) = -\sin\theta$	$\sin(\pi - \theta) = \sin\theta$	$\sin(2\pi - \theta) = -\sin\theta$
$\cos(\theta + \frac{\pi}{2}) = -\sin\theta$	$\cos(\theta + \pi) = -\cos\theta$	$\cos(\pi - \theta) = -\cos\theta$	$\cos(2\pi - \theta) = \cos\theta$
$\tan(\theta + \frac{\pi}{2}) = -\cot\theta$	$\tan(\theta + \pi) = +\tan\theta$	$\tan(\pi - \theta) = -\tan\theta$	$\tan(2\pi - \theta) = -\tan\theta$
$\cot(\theta + \frac{\pi}{2}) = -\tan\theta$	$\cot(\theta + \pi) = +\cot\theta$	$\cot(\pi - \theta) = -\cot\theta$	$\cot(2\pi - \theta) = -\cot\theta$
$\sec(\theta + \frac{\pi}{2}) = -\csc\theta$	$\sec(\theta + \pi) = -\sec\theta$	$\sec(\pi - \theta) = -\sec\theta$	$\sec(2\pi - \theta) = \sec\theta$
$\csc(\theta + \frac{\pi}{2}) = +\sec\theta$	$\csc(\theta + \pi) = -\csc\theta$	$\csc(\pi - \theta) = \csc\theta$	$\csc(2\pi - \theta) = -\csc\theta$

Pythagorean trigonometric identity: $\sin^2 x + \cos^2 x = 1$.

Special values:

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin t$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos t$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Addition formulas:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y,$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y.$$

Double-angle formulas:

$$\sin 2x = 2 \sin x \cos x, \quad \cos 2x = \cos^2 x - \sin^2 x.$$

Half-angle formula.

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}.$$

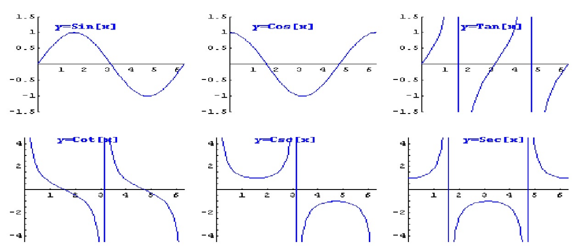
Periods: $\sin x$ and $\cos x$ have period 2π, $\tan x$ and $\cot x$ have period π.

Example 16 *Find all values of x in the interval $[0, 2\pi]$ such that $\sin^2 x - 3\cos^2 x = 0$.*

Example 17 *Find $\cos x$ where $x \in [\frac{\pi}{2}, 2\pi]$ such that $\sin x = 0.8$.*

Solution: $\cos x = -0.6$

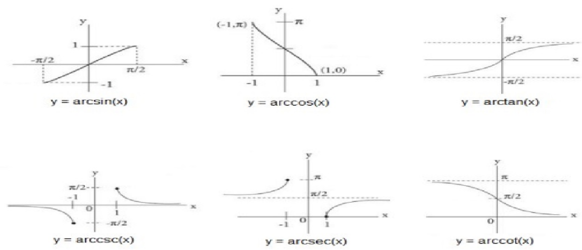
Graphs.



3.6 Inverse Trig Functions

Inverse Trig Function	Domain	Restriction (Range)	Meaning
$y = \arcsin x = \sin^{-1}(x)$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$	$\sin y = x$
$y = \arccos x = \cos^{-1}(x)$	$-1 \le x \le 1$	$0 \le y \le \pi$	$\cos y = x$
$y = \arctan x = \tan^{-1}(x)$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$	$\tan y = x$
$y = arcsec\ x = \sec^{-1}(x)$	$\|x\| \ge 1$	$0 \le y \le \pi, y \ne \frac{\pi}{2}$	$\sec y = x$
$y = arccsc\ x = \csc^{-1}(x)$	$\|x\| \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2},\ y \ne 0$	$\csc y = x$
$y = arccot\ x = \cot^{-1}(x)$	$-\infty < x < \infty$	$0 < y < \pi$	$\tan y = x$

Graphs of the inverse functions: Using the symmetry line $y = x$ to get the graph for inverse from original functions.



Example 18 *Find the exact values of the following expressions: (a)* $\arcsin(1)$ *(b)* $\arctan(-1)$ *(c)* $\tan^{-1}(\sqrt{3})$ *(d)*$\sin[\cos^{-1}(\frac{\sqrt{3}}{2})]$ *(e)* $\arctan(\tan x)$*, where* $\frac{3\pi}{4} \leq x \leq 2\pi$*.*

Example 19 *Simplify the following expression:* $\tan \arcsin \frac{x}{a}$.

Chapter 2 – Limits and Derivatives

2.1 The Tangent and Velocity Problem

The average rate of change of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, h \neq 0.$$

Geometrically, it is the slope of the secant through two points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Instantaneous rates of change and tangent lines: What is a tangent line at point P on a curve? We chose another point Q on the curve. The line PQ is called a secant line. When Q tends to P, the secant PQ will tends to a line, which is called a the tangent line of the curve at P.

Example 20 *Estimate the slope of the tangent line to the parabola* $y = x^2$ *at the point* $(2, 4)$.

Solution:

$$m_{sec} = \frac{x^2 - 4}{x - 2}.$$

x	m
2.1	4.1
2.01	4.01
2.001	4.001
1.9	3.9
1.99	3.99
1.999	3.999

Definition 1 *Let* $s = f(t)$ *be position function.*

$$\textit{average velocity} = \frac{\textit{total distance}}{\textit{total time}} = \frac{\Delta s}{\Delta t}.$$

Example 21 *Consider the position function* $s = t^2 - 3t + 5$*. Find the average velocity from* $t = 3$ *to* $t = 4$*.*

2.2 The Limit of A Function

Definition 2 *We write*

$$f(a-0) = \lim_{x \to a^-} f(x) = L$$

and say that the limit of $f(x)$ *is* L *as* x *approaches* a *from the left. Similarly, We write*

$$f(a+0) = \lim_{x \to a^+} f(x) = L$$

and say that the limit of $f(x)$ *is* L *as* x *approaches* a *from the right.*

Example 22 *Consider the Heaviside function*

$$H(t) = \begin{cases} 0, & t < 0; \\ 1, & t \geq 0. \end{cases}$$

$$\lim_{t \to 2^-} H(t) = 1,$$

$$\lim_{t \to 0^+} H(t) = 1, \lim_{t \to 0^-} H(t) = 0.$$

Example 23 $\lim_{x \to 0^-} \frac{|x|}{x} = -1, \lim_{x \to 0^+} \frac{|x|}{x} = 1.$

Example 24 *Let*

$$f(x) = \begin{cases} x - 5, & x < 0; \\ x^2 + 3x, & 0 \leq x \leq 1; \\ x^4 - x^3 + 4, & x > 1. \end{cases}$$

Then $\lim_{x \to 0^-} f(x) = -5$ *and* $\lim_{x \to 1^+} f(x) = 4$*.*

Definition 3 *We write*

$$\lim_{x \to a} f(x) = L$$

and say "as x *approaches* a*, the limit of* $f(x)$ *is* L*." If* L *is a finite number, we say that the limit exists, otherwise, the limit does not exist.*

Theorem 1

$$\lim_{x\to a} f(x) = L \Leftrightarrow \lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = L.$$

Example 25 *Consider the Heaviside function*

$$H(t) = \begin{cases} 0, & t < 0; \\ 1, & t \geq 0. \end{cases}$$

$$\lim_{t\to 2} H(t) = 1,$$

$$\lim_{t\to 0^+} H(t) = 1, \lim_{t\to 0^-} H(t) = 0, \Rightarrow \lim_{t\to 0} H(t) \nexists.$$

Example 26 $\lim_{x\to 0} \frac{|x|}{x} \nexists$.

$\because \lim_{x\to 0^-} \frac{|x|}{x} = -1, \lim_{x\to 0^+} \frac{|x|}{x} = 1.$

Example 27 *Let*

$$f(x) = \begin{cases} x - 5, & x < 0; \\ x^2 + 3x, & 0 \leq x \leq 1; \\ x^4 - x^3 + 4, & x > 1. \end{cases}$$

Then $\lim_{x\to 0} f(x) \nexists$ *and* $\lim_{x\to 1} f(x) = 4$.

Euler's Number e

$$e = \lim_{x\to 0}(1 + x)^{1/x} = \lim_{x\to\infty}(1 + \frac{1}{x})^x = 2.71828....$$

Example 28 *Calculate*

$$\lim_{x\to 0}(1 - x)^{1/x}, \quad \lim_{x\to\infty}(1 + \frac{3}{x})^x.$$

Example 29 *Evaluate*

$$\lim_{x\to 0^-} e^{1/x}.$$

Infinite Limits: Vertical Asymptote

Definition 4

$$\lim_{x\to a} f(x) = \infty$$

means that $f(x)$ *can be arbitrarily large as* x *tends to* a*;*

$$\lim_{x\to a} f(x) = -\infty$$

means that $f(x)$ *can be arbitrarily large negative as* x *tends to* a.

Example 30 $\lim_{x\to 0}\frac{1}{x^2}=\infty$, $\lim_{x\to 1}\frac{-1}{(x-1)^2}=-\infty$.

Definition 5 *The line $x=a$ is called a vertical asymptote of the curve $y=f(x)$ if at least one of the following statements is true:*

$$\lim_{x\to a^-} f(x)=\pm\infty,\ \lim_{x\to a^+} f(x)=\pm\infty, \lim_{x\to a} f(x)=\pm\infty.$$

Example 31 *Find VA:* $f(x)=\tan x$, $\ln x$.

2.3 Calculating Limits using The Limit Laws

Properties: Suppose that $\lim\limits_{x\to a} f(x)\ \exists$ and $\lim\limits_{x\to a} g(x)\ \exists$.

- $\lim\limits_{x\to a} P(x)=P(a)$, $\quad P(x)$ is a polynomial.
- $\lim\limits_{x\to a}(cf(x)\pm dg(x))=c\lim\limits_{x\to a} f(x)\pm d\lim\limits_{x\to a} g(x)$, $\quad$ c, d are constants.
- $\lim\limits_{x\to a}[f(x)g(x)]=\lim\limits_{x\to a} f(x)\cdot\lim\limits_{x\to a} g(x)$.
- $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, $\quad if\ \lim\limits_{x\to a} g(x)\neq 0$.
- $\lim\limits_{x\to a}[f(x)]^n=[\lim\limits_{x\to a} f(x)]^n$.
- $\lim\limits_{x\to a}\sqrt[n]{f(x)}=\sqrt[n]{\lim\limits_{x\to a} f(x)}$. When n is even, we assume that $\lim\limits_{x\to a} f(x)\neq 0$.

Example 32

$$\lim_{x\to 1}(x^2-3)=1^2-3=-2,\quad \lim_{x\to 1}\frac{3x^4+8x-2}{x-2}=\frac{3(1)^4+8(1)-2}{1-2}=-9.$$

Special case:

$$\lim_{x\to a}\frac{f(x)}{g(x)}\quad where\ g(a)=0.$$

- If $f(a)\neq 0$, then $\lim_{x\to a}\frac{f(x)}{g(x)}$ does not exist.

- If $f(a) = 0$, then simplify $\frac{f(x)}{g(x)}$ first, then study the limit.

Example 33

$$\lim_{x\to 2} \frac{3x^4 + 8x - 2}{x-2} \nexists, \quad \lim_{x\to 2} \frac{x-2}{x-2} = 1.$$

Example 34

$$\begin{aligned}
\lim_{x\to 2} \frac{x^2-4}{x-2} &= \lim_{x\to 2}(x+2) = 4, \\
\lim_{h\to 0} \frac{(h+1)^2 - 1}{h} &= \lim_{h\to 0} \frac{h(h+2)}{h} = \lim_{h\to 0}(h+2) = 2, \\
\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x} &= \lim_{x\to 0} \frac{(\sqrt{x+4}-2)(\sqrt{x+4}+2)}{x(\sqrt{x+4}+2)} = \lim_{x\to 0} \frac{x}{x(\sqrt{x+4}+2)} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{x+4}+2} = \frac{1}{4}.
\end{aligned}$$

Theorem 2 *If $f(x) \le g(x)$ near $x = a$, then*

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x).$$

Theorem 3 *The Sandwich Theorem (The Squeeze Theorem): If $f(x) \le g(x) \le h(x)$ near $x = a$, and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.*

Example 35 *Show that*

$$\lim_{x\to 0} x^4 \cos\frac{3}{x} = 0$$

by the Squeeze Theorem.

Example 36

$$\lim_{x\to 0} \sin x = 0, \quad \lim_{x\to 0} \cos x = 1.$$

Example 37 *Estimate the limit of*

$$\lim_{x\to 0} \frac{\sin x}{x}.$$

x	$\frac{\sin x}{x}$
1	0.84147098
0.1	0.99833417
0.01	0.99998333
0.001	0.99999983

Famous result:

$$\lim_{h\to 0} \frac{\sin h}{h} = 1.$$

Proof. It is from the inequality

$$\cos x < \frac{\sin x}{x} < 1.$$

This will imply that

$$\lim_{h\to 0}\frac{\cos h - 1}{h} = \lim_{h\to 0}\frac{\cos^2 h - 1}{h(\cos h + 1)} = \lim_{h\to 0}\frac{\sin h}{h}\frac{\sin h}{\cos h + 1} = 0.$$

Example 38

$$\lim_{x\to 0}\frac{\sin 2x}{\sin 3x} = \lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\frac{3x}{\sin 3x}\cdot\frac{2x}{3x} = \frac{2}{3}.$$

2.5 Continuity

Definition 6 *If $\lim_{x\to a} f(x) = f(a)$, then $f(x)$ is continuous at $x = a$, otherwise, $f(x)$ is discontinuous at $x = a$. If $f(x)$ is continuous at any point on an interval, then $f(x)$ is continuous on the interval. For the end points, we only need sided limits.*

Example 39 *Explore discontinuity from graph.*

Example 40 *Consider $f(x) = \frac{x^2-2x+1}{x-1}$ at $x = 1$. $f(x)$ is undefined at $x = 1$. But $\lim_{x\to 1} f(x) = 0$. So the discontinuous point $x = 1$ is* **removable** *if we define $f(1) = 0$.*

Example 41 *Determine the continuity of $f(x) = \frac{|x|}{x}$.*

Definition 7 *If $\lim_{x\to a^-} f(x) = f(a)$, then $f(x)$ is continuous from the left at $x = a$; if $\lim_{x\to a^+} f(x) = f(a)$, then $f(x)$ is continuous from the right at $x = a$.*

Example 42 *Determine the left and right continuity at $x = 0$:*

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0; \\ 1, & x = 0. \end{cases}.$$

Theorem 4 *If $f(x)$ and $g(x)$ are continuous at a, then*

$$f \pm g, fg, cf\,(c \text{ is a constant}), \frac{f}{g}\,(\text{if } g(a) \neq 0)$$

are continuous.

Theorem 5 *Polynomials, rational functions, root functions, trig functions, inverse trig functions, exponential functions and logarithmic functions are continuous in their domain.*

Theorem 6 *If* $\lim_{x\to a} g(x) = b$ *and* $f(x)$ *is continuous at* b*, then*

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b).$$

Furthermore, if $g(x)$ *is continuous at* a*, and* $f(x)$ *is continuous at* $g(a)$*, then* $f(g(x))$ *is continuous at* a*.*

Example 43 *Find* k *such that* $f(x) = \begin{cases} x^3 + kx^2 - 5x, & x > 2; \\ \frac{x}{x-3}, & x \leq 2 \end{cases}$ *is continuous at* $x = 2$*.*

Example 44 *The greatest integer function [x].*

Theorem 7 *(The Intermediate Theorem) If* $f(x)$ *is continuous on* $[a, b]$*, and* y_0 *between* $f(a)$ *and* $f(b)$*, then* $\exists c \in [a, b]$ *such that* $f(c) = y_0$*.*

Example 45

$$\frac{4}{x-5} + \frac{-9x}{(x+1)(x-2)(x+3)} = 0$$

has solutions in $(0, 1)$*.*

Proof. Let $f(x) = \frac{4}{x-5} + \frac{-9x}{(x+1)(x-2)(x+3)}$. Then $f(x)$ is continuous on [0,1]. $f(0) = -4/5$, $f(1) = 1/8$. The conclusion follows from The Intermediate Theorem.

Example 46

$$\lim_{x\to 1} \arcsin\left(\frac{1-\sqrt{x}}{1-x}\right) = \arcsin\left(\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}\right) = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}.$$

2.6 Limits at Infinity, HA

Definition 8 *The line* $y = L$ *is called a horizontal asymptote of the curve* $y = f(x)$ *if either*

$$\lim_{x\to\infty} f(x) = L \text{ or } \lim_{x\to -\infty} f(x) = L.$$

Example 47 $f(x) = \frac{3x^2-x-1}{2x^2+3x}$ *has horizontal asymptote* $y = \frac{3}{2}$*.*

Example 48 $\lim_{x\to\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0} = \begin{cases} 0, & \text{if } n < m; \\ \frac{a_n}{b_n}, & \text{if } n = m; \\ \pm\infty, & \text{if } n > m. \end{cases}$.

Example 49 $\lim_{x\to\infty} \sin x$, $\lim_{x\to\infty} \cos x$ *do not exist.*

Example 50 *Find the horizontal asymptotes of the function* $f(x) = e^x$.

Sol: $\lim_{x\to-\infty} e^x = 0$. Thus, HA: $y = 0$.

Example 51 *Find the horizontal asymptotes of the function*

$$f(x) = \sqrt{x^2 + 1} - x.$$

Example 52 *Find the horizontal asymptotes of the function*

$$f(x) = \sqrt{x^2 + 5x + 1} - x.$$

Example 53 $y = \tan^{-1} x$ *has horizontal asymptotes* $y = \frac{\pi}{2}$ *and* $y = -\frac{\pi}{2}$.

2.7 Derivatives and Rates of Change

Definition 9 *Let $P = (a, f(a))$ be a point on the curve $y = f(x)$. The tangent of $f(x)$ at P is the line through P with slope*

$$m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Example 54 *Find the slope and the equation of the tangent line to the curve*

$$y = f(x) = 3x^2 - 6x + 1$$

at the point $(2, 1)$. Sketch the curve.

Example 55 *Find the tangent line to the hyperbola $xy = 4$ at the point $(1, 4)$.*

Example 56 *Find the slope of the tangent line to the curve $y = \frac{1}{\sqrt{x+1}}$ at the point $(0, 1)$.*

Definition 10 *Let $s = f(t)$ be position function.*

$$\textit{average velocity} = \frac{\textit{total distance}}{\textit{total time}} = \frac{\Delta s}{\Delta t}.$$

Instantaneous velocity, or velocity, or rate of change at $t = a$ is

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Example 57 *Consider the position function*

$$s = t^2 - 3t + 5.$$

Find the velocity at $t = 1$ and $t = 4$, interpret your results.

Definition 11 *The derivative of the function $y = f(x)$ at a is*

$$y'(a) = \frac{dy}{dx} = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Meaning of $f'(a)$:

- instantaneous rate of change of $f(x)$ at a, or

- rate of change of $f(x)$ at a, or
- the slope of the tangent line to the curve at a.

Example 58 *Let* $f(x) = x^2$. *Calculate* $f'(5)$.

Example 59 *The volume of a sphere of radius* r *is given by*

$$V = \frac{4}{3}\pi r^3.$$

Calculate $\frac{dV}{dr}$ *by definition. What's the meaning of this derivative?*

Example 60 *A spherical balloon is being inflated. Find the rate of change of the volume with respect to the radius when the radius is 2cm.*

2.8 The derivative as a function

Definition 12 *The derivative of the function $y = f(x)$ is the function $f'(x)$:*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Example 61 *Let $f(x) = \sqrt{x-3}$. Find $f'(x)$ and state the domains of f and f'.*

Example 62 *Find $f'(x)$ from the graph of f.*

Definition 13 *The function f is differentiable at a if $f'(a)$ exists. It is differentiable on an interval if $f'(a)$ exists for any a on the interval.*

Theorem 8 *If a function is differentiable at $x = c$, then the function is continuous at $x = c$.*

Example 63 *$f(x) = |x|$ is not differentiable at $x = 0$.*

Higher derivatives:

- Let $y = f(x)$. Then

$$y''(x) = f''(x) = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dx}\right), \quad y^{(n)}(x) = f^{(n)}(x) = \frac{d}{dx}\left(\frac{dy^{(n-1)}}{dx}\right).$$

- If $s(t)$ is a position function, then the velocity is $v(t) = s'(t)$, acceleration is $a(t) = v'(t) = s''(t)$.

Example 64 *Let $f(x) = 4x^3 + 6x^2 - 23x + 7$. Then $f''(x) = 24x + 12$, $f'''(x) = 24$ and $f^{(4)}(x) = 0$.*

Example 65 *Let $f(x) = (x^2 - x - 1)^{100}$. Calculate $f''(x)$.*

Example 66

$$(x^n)^{(n)} = n!, \quad (\frac{1}{x})^{(n)} = (-1)^n n! x^{-n-1}.$$

Example 67 *The position of a particle is given by*

$$s = t^3 - 15t^2 + 63t, \quad t \geq 0$$

where s is measured in meters and t in seconds.

a) What is the initial position? initial velocity? initial acceleration?

b) Find the velocity after 1s and 4s.

c) When is the particle at rest?

d) When is the particle moving in the positive direction?

e) When is the acceleration 0?

f) Find the displacement and the velocity at that time from e).

Chapter 3. Differentiation Rules

3.1 Derivatives of Polynomials and Exponential Functions

- Constant rule: If $f(x) = c$, then $f'(x) = 0$ or $\frac{d}{dx}(c) = 0$.
- Power Rule: If $f(x) = x^n$, n is any real number. Then $f'(x) = nx^{n-1}$.
- Constant multiple rule: $[cf(x)]' = cf'(x)$.
- Sum rule and difference rule: $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$
- Derivative of polynomial: $[a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0]' = a_n n x^{n-1} + a_{n-1}(n-1)x^{n-2} + ... + a_1$.
- Derivative of exponential function:

$$(e^x)' = e^x.$$

Example 68 *Let* $f(x) = 4x^3 + 6x^2 - 23x + 7$. *Find the equation of the tangent line at* $(1, -6)$.

Example 69 *At what point(s) on the curve* $y = e^x$ *is the tangent line*

a) parallel to $y = 3x - 2$*?*

Solution: $(\ln 3, 3)$.

b) perpendicular to $y = -2x$*?*

Solution: $(-\ln 2, 1/2)$.

3.2 The product and quotient rules

- Product rule:

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x).$$

- Quotient rule:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Example. Let $f(x) = \sqrt{x}e^x$. Calculate $f'(4)$.
Example. Let $f(x) = \frac{\sqrt{x}+x^2}{e^x+x}$. Calculate $f'(4)$.
Example. Let $f(x) = \frac{x^3+4x^2}{x^5+x+1}$. Calculate $f'(1)$.

Example 70 *Let* $f(x) = \frac{x}{e^x}$. *Calculate* $f^{(n)}(x)$.

Example 71 *At what point(s) on the curve* $y = \frac{x^2-4}{x+1}$ *is the tangent line*
a) parallel to $y = 3x$*?*
b) perpendicular to $y = -0.5x$*?*

3.3 Derivatives of Trigonometric Functions

Famous result:

$$\lim_{h\to 0} \frac{\sin h}{h} = 1.$$

This will imply that

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)} = \lim_{h\to 0} \frac{\sin h}{h} \frac{\sin h}{\cos h + 1} = 0.$$

Derivative of Trig Functions:

$$(\sin x)' = \cos x, \quad (\cos x)' = -\sin x, \quad (\tan x)' = \sec^2 x,$$

$$(\sec x)' = \sec x \tan x, \quad (\csc x)' = -\csc x \cot x, \quad (\cot x)' = -\csc^2 x.$$

Example 72 *Differentiate* $\csc x$, $\cot x$, $e^x \cos(x)$, $\frac{1+\cos x}{1+\sin x}$, $e^x \sin x$.

Example 73 *Let* $y = \sin(x)$*, calculate* $y^{(10)}(x)$*.*

Example 74 *Given the position function* $s = f(t) = 2\sin(t)$*, calculate the velocity and acceleration at* $t = \frac{\pi}{3}$*.*

Example 75 *Find the equation of the tangent line to the curve* $\sin(1-x)$ *at* $(1, 0)$*.*

Hint: use the definition of the derivative of the function.

3.4 The chain rule

- Chain Rule:

$$[f(g(x))]' = f'(g(x))g'(x), \quad \frac{df(g(x))}{dx} = \frac{df(v)}{dv} \cdot \frac{dg(x)}{dx}, v = g(x), \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

- General Power Rule:

$$[u(x)^n]' = nu^{n-1}u'(x).$$

Example 76 *Let* $f(x) = (x^2 - x - 1)^{100}$*. Calculate* $f'(x)$*.*

Example 77 *Let* $h(x) = g(f(x))$*, where* $f'(2) = 3$*,* $f(2) = 4$*,* $g'(3) = -5$*,* $g(4) = 8$*,* $g'(4) = 7$*. Find* $h'(2)$*.*

Example 78 *Let* $y = \sqrt{x + \sqrt{x^2 + x}}$*. Calculate* y'*.*

Example 79 *Find* $f'(x)$*. If*

$$f(x) = \sin x^2, \quad \sin^2 x, \quad e^{\sin x}, \quad \sin(\cos(\tan x))).$$

Derivative of exponential functions:

$$(a^x)' = a^x \ln a.$$

Proof.

$$(a^x)' = (e^{\ln a^x})' = (e^{x \ln a})' = (e^{x \ln a})(x \ln a)' = a^x \ln a.$$

3.5 Implicit differentiation

Implicit Differentiation: Assume $f(x, y) = C$. To find y',

- consider x as an independent variable, y as a dependent variable;
- differentiate both sides with respect to x;
- isolate y'.

Example 80 *Find y' from $y^2 + x^2 = 1$.*

Example 81 *Let*

$$y^2 + x^2 = xy + 3.$$

1) Find the equation of the tangent line to the curve at $(0, \sqrt{3})$.
2) Find all the points on the curve where the tangent line is either horizontal or vertical.

Example 82 *Find y' from $\tan(xy + x) = x + y$.*

Derivatives of Inverse Trig Functions

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}\arccos x = \frac{1}{-\sqrt{1-x^2}}, \quad \frac{d}{dx}\arctan x = \frac{1}{1+x^2},$$

$$\frac{d}{dx}\text{arcsec}x = \frac{1}{|x|\sqrt{x^2-1}}, \quad \frac{d}{dx}\text{arccsc}x = -\frac{1}{|x|\sqrt{x^2-1}}, \quad \frac{d}{dx}\text{arccot}x = -\frac{1}{1+x^2}.$$

Example 83 $y = \sin(\arctan 2x)$, $y = \arcsin\left(\frac{b+a\cos x}{a+b\cos x}\right)$.

3.6 Derivative of Logarithmic Function

By using the formula

$$\frac{df^{-1}(x)}{dx} = \frac{1}{f'(f^{-1}(x))},$$

We can get some special results:

- Derivatives of log functions:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad (\ln f(x))' = \frac{f'(x)}{f(x)},$$

$$(\log_a |x|)' = \frac{1}{x \ln a}, \quad (\log_a f(x))' = \frac{f'(x)}{f(x) \ln a}, \ldots$$

Change base:

$$\log_a b = \frac{\log_c b}{\log_c a}.$$

Example 84 *Differentiate* $\ln(x^2 + 1)$.

Logarithmic differentiation

Example 85 *Differentiate* $y = \frac{(x^2+x+5)\arcsin x}{(x+1)^2}$.

Example 86 *Differentiate* x^x, $(\sin x)^x$.

Number e

$$e = \lim_{x \to 0}(1 + x)^{1/x} = \lim_{x \to \infty}(1 + \frac{1}{x})^x.$$

3.9 Related Rates

Let $y = f(x)$. Then $f'(x)$ measures how fast $f(x)$ is increasing or decreasing.

Solving a Related Rates Problem

Step 1: Identify the changing quantities.

Step 2: Construct an equation that relates the changing quantities.

Step 3: Differentiate both sides of the equation with respect to the third variable.

Step 4: Isolate the required quantity, or substitute the given values in the derived equation you obtained above, and solve for the required quantity.

Example 87 *The area of a circle is growing at a rate of* $12cm^2/s$. *How fast is the radius growing when the radius equals 10 cm?*

Example 88 *A 13-foot ladder leaning against the wall starts to slide down the wall at a rate of 4 ft/min. How fast is the base of the ladder moving when it is 5 feet from the wall?*

3.10 Linear Approximation and Differentials

LINEAR APPROXIMATIONS: we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a.

Definition 14 *The approximation*

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the linear approximation or tangent line approximation of f at a.

$$L(x) = f(a) + f'(a)(x - a)$$

is called the linearization of f at a.

Example 89 *a) Find the linearization of the function $f(x) = \sqrt{x}$ at $a = 9$ and use it to approximate the numbers $\sqrt{9.01}$.*

b) Are these approximations overestimates or underestimates?

c) For what values of x is the linear approximation in a) accurate to within 0.1?

Example 90 *The linearization of the function $f(x) = \sin x$ at $a = 0$ is $L(x) = x$.*

Definition 15 *If $y = f(x)$, where f is a differentiable function, then the differential dx is an independent variable. That is, dx can be given the value of any real number. The differential dy is then defined in terms of dx by the equation $dy = f'(x)dx$.*

Remark. dy represents the amount that the tangent line rises or falls (the change in the linearization). Δy represents the amount that the curve y = f(x) rises or falls when changes by an amount dx.

Example 91 *Compare the values of Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes from: 2 to 2.01.*

Remark. In the notation of differentials, the linear approximation can be written as:

$$f(a + dx) \approx f(a) + dy.$$

Example 92 *The radius of a sphere was measured to be 21 cm with a possible error of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?*

4.9 and 5.4 (Part I) Antiderivatives

Definition 16 *A function $F(x)$ is called an antiderivative of $f(x)$ on an interval I if $F'(x) = f(x)$ for all x in I.*

Some basic results:

function	antiderivative	formula
k	$kx + C$	$\int k dx = kx + C$
$x^n, n \neq -1$	$\frac{x^{n+1}}{n+1} + C$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C;\ (n \neq -1)$
e^{kx}	$\frac{1}{k} e^{kx} + C$	$\int e^{kx} dx = \frac{1}{k} e^{kx} + C$
a^{kx}	$\frac{a^{kx}}{k \ln a} + C$	$\int a^{kx} dx = \frac{a^{kx}}{k \ln a} + C$
$\frac{1}{x}$	$\ln \lvert x \rvert + C$	$\int \frac{1}{x} dx = \ln \lvert x \rvert + C$
$\cos kx$	$\frac{1}{k} \sin kx + C$	$\int \cos kx dx = \frac{1}{k} \sin kx + C$
$\sin kx$	$-\frac{1}{k} \cos kx + C$	$\int \sin kx dx = -\frac{1}{k} \cos kx + C$
$\sec^2 kx$	$\frac{1}{k} \tan kx + C$	$\int \sec^2 kx dx = \frac{1}{k} \tan kx + C$
$\sec kx \tan kx$	$\frac{1}{k} \sec kx + C$	$\int \sec kx \tan kx dx = \frac{1}{k} \sec kx + C$
$\frac{1}{\sqrt{1-(kx)^2}}$	$\frac{1}{k} \arcsin kx + C$	$\int \frac{1}{\sqrt{1-(kx)^2}} dx = \frac{1}{k} \arcsin kx + C$
$\frac{1}{1+(kx)^2}$	$\frac{1}{k} \arctan kx + C$	$\int \frac{1}{1+(kx)^2} dx = \frac{1}{k} \arctan kx + C$
		$\int k f(x) dx = k \int f(x) dx$
		$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx.$

Example 93 $\int \frac{x^2-1}{x^3} dx = \int (x^{-1} - x^{-3}) dx = \ln \lvert x \rvert + \frac{1}{2x^2} + C.$

$\int \sin 4x + e^{5x} dx = -\frac{1}{4} \cos 4x + \frac{1}{5} e^{5x} + C.$

Example 94 *Find $f(x)$ such that*

$$f'(x) = \sin x + \frac{4x^2 - 22}{x^3}.$$

Example 95 *Find $f(x)$ such that*

$$f'(x) = \sin x + \frac{4x^2 - 22}{x^3}, \quad f(1) = 3.$$

Example 96 *Let $y = y(x)$ be the solution of the differential equation*

$$\frac{dy}{dx} = \frac{\pi \sin \pi x + 4x}{9y^2}$$

subject to the initial condition $y(0) = 1$. Find $y(1)$.

5.1 Area and Distances

We need three formulas for calculating the sum:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Three ways to estimate the area of the region S bounded by the continuous function $y = f(x)$ (where $f(x) \geq 0$), $x = a$, $x = b$ and the x-axis:

We divide the interval $[a, b]$ into n equal parts with endpoints $x_0 = a$, $x_1 = a + \frac{b-a}{n}$, $x_2 = a + \frac{2(b-a)}{n}$,..., $x_n = a + \frac{n(b-a)}{n} = b$, $\Delta x = \frac{b-a}{n}$,

$$L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x = [f(x_0) + f(x_1) + ... + f(x_{n-1})]\,\Delta x,$$
$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = [f(x_1) + ... + f(x_{n-1}) + f(x_n)]\,\Delta x,$$
$$M_n = \sum_{i=1}^{n} f(\frac{x_{i-1}+x_i}{2})\Delta x = \left[f(\frac{x_0+x_1}{2}) + f(\frac{x_1+x_2}{2}) + ... + f(\frac{x_{n-1}+x_n}{2})\right]\Delta x.$$

Here L_n is called Left-hand Sum, R_n is Right-hand Sum, M_n is called Midpoint Sum, or Midpoint Rule.

Example 97 *Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.*

Definition 17 *The area under the curve $y = f(x)$ between $x = a$ and $x = b$ is:*

$$Area = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \lim_{n\to\infty} M_n.$$

Now, let's consider the distance problem:

Find the distance traveled by an object during a certain time period if the velocity of the object is known at all times.

If the velocity remains constant, then

Distance=(velocity)(time).

For given velocity function $y = v(t)$, total distance from $t = a$ to $t = b$ is the area under the curve $y = v(t)$.

$$d = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n,$$

where we replace x by t and f by v in the definition.

5.2 The Definite Integral

Definition 18 *Definite integral = limit of Riemann sum:*

$$\int_a^b f(x)dx = \lim_{n\to\infty} GRS,$$

where

$$GRS(\textit{General Riemann Sum}) = \sum_{i=1}^{n} f(c_i)\Delta x, \quad x_{i-1} \le c_i \le x_i, \Delta x = \frac{b-a}{n}.$$

The relation to area is:

$$\int_a^b f(x)dx = \textit{area above } x\textit{-axis - area below } x\textit{-axis.}$$

Example 98 *Estimate*

$$\int_0^2 \frac{1}{1+t^2}dt$$

by L_n *with* $\Delta t = 1$ *and* $\Delta t = 0.5$.

Example 99 *Calculate* $\int_0^5 f(x)dx$, *where*

$$f(x) = \begin{cases} x, & 0 \le x \le 1; \\ 1, & 1 \le x \le 2; \\ 3-x, & 2 \le x \le 3; \\ -\sqrt{1-(x-4)^2}, & 3 \le x \le 5. \end{cases}$$

Some basic properties about definite integral:

- $\int_a^b c dx = c(b-a)$;
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$;
- $\int_a^a f(x)dx = 0$;
- $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$;
- $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$;
- Constant multiple: $\int_a^b cf(x)dx = c\int_a^b f(x)dx$;
- Comparison of Definite Integrals: If $f(x) \leq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

Example 100 *Let* $\int_1^5 f(x)\,dx = 3$, $\int_1^5 g(x)\,dx = 5$. *Calculate* $\int_1^5 [2f(x) - g(x) - 1]\,dx$.

Example 101 *Find an upper bound and a lower bound to*

$$\int_{\pi/3}^{5\pi/6} \sin x\, dx.$$

5.3 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus :

- **If** $F'(x) = f(x)$**, then**
$$\int_a^b f(x)dx = F(b) - F(a).$$
- **If**
$$g(x) = \int_a^x f(t)dt,$$
then $g'(x) = f(x)$**.**
- **General formula**
$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)dt = b'(x)f(b(x)) - a'(x)f(a(x)).$$

Example 102

$$\begin{aligned}
\frac{d}{dx}\int_0^x f(t)dt &= f(x), \\
\frac{d}{dx}\int_0^{x^2} f(t)dt &= \frac{d}{du}\int_0^u f(t)dt \cdot \frac{du}{dx} = 2xf(x^2), \quad u = x^2, \\
\frac{d}{dx}\int_{x^2}^{x^3} f(t)dt &= \frac{d}{dx}\left(\int_{x^2}^0 f(t)dt + \int_0^{x^3} f(t)dt\right) = -2xf(x^2) + 3x^2 f(x^3).
\end{aligned}$$

Example 103 ***Let*** $g(x) = \int_a^x e^{t^2}dt$***. Calculate*** $g'(2)$ ***and*** $g''(2)$***.***

Example 104 ***Calculate*** $\int_0^2 3^t dt$.

Average value of $f(x)$ **from** a **to** b **=**

$$\frac{1}{b-a}\int_a^b f(x)dx.$$

Example. Find the average value of x^3 **over** $[0, 2]$**.**

5.4 Indefinite Integrals and The Net Change Theorem

Definition 19 ***Antiderivative of*** $f(x)$ ***is denoted by*** $\int f(x)dx$ ***, which is called an indefinite integral.***

Applications:

1. **Net Change = The integral of a rate of change =** $\int_a^b F'(x)dx = F(b) - F(a)$**.**

2. **If** $v(t)$ **is the volume of water in a reservoir at time t, its derivative** $v'(t)$ **is the rate at which water flows into the reservoir at time t. The change in the amount of water in the reservoir between time** $t = a$ **and time** $t = b$ **is:**

$$\int_a^b v'(t)dt = v(b) - v(a).$$

3. **If the mass of a rod measured from the left end to a point x is $m(x)$, then the linear density is $\rho(x) = m'(x)$. the mass of the segment of the rod that lies between x = a and x = b is:**

$$\int_a^b \rho(x)dx = m(b) - m(a).$$

4. **If an object moves along a straight line with position function s(t), then its velocity is $v(t) = s'(t)$. Then**

   the total distance traveled between $t = a$ and $t = b$ is: $\int_a^b |v(t)|dt$.

Example 105 ***Let $f'(x) = x^3$, $f(0) = 1$. Calculate $f(2)$.***

Example 106 ***An object moves along a coordinate line with velocity***

$$v(t) = 2 - 3t + t^2 \quad \textit{units/s}.$$

Its initial position is 2 units to the right of the origin(when t=0). Find the position of the object and total distance it traveled 4s later.

Example 107 ***Find the area of the region between the x-axis and the graph of $f(x) = x^3 - 2x^2 - 3x$, $-1 \le x \le 3$.***

5.5 The Substitution Rule

The substitution rules:

- **For indefinite integral:** $\int f(g(x))g'(x)dx = \int f(u)du, \quad u = g(x)$. **In the final result, we have to replace** u **by** $g(x)$**;**
- **For definite integral:** $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$

Example 108 ***Evaluate***

$$\int (2x-1)(x^2-x)^{100}\,dx.$$

Example 109

$$\int x\sqrt{x^2+1}\,dx.$$

Example 110

$$\int_0^1 x\sqrt{x^2+1}\,dx.$$

Example 111 ***Find***

$$\int \frac{1}{e^{-x}+1}dx.$$

Example 112 ***Evaluate***

$$\int x^2 e^{x^3+1}\,dx.$$

Example 113 ***Evaluate***

$$\int \frac{x}{\sqrt{1-x}}\,dx.$$

Example 114 ***Calculate***

$$\int \tan x\,dx.$$

7.1 Integration by Parts

Integration by parts:

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx.$$

Example 115 ***Evaluate***

$$\int (x^2 + x + 1)e^x \, dx.$$

Example 116 ***Evaluate***

$$\int 4x^3 \ln x \, dx.$$

Example 117 ***Evaluate***

$$\int_0^1 \arctan x \, dx.$$

Example 118 ***Evaluate***

$$\int e^x \cos x \, dx.$$

7.2 Trigonometric Integrals

Method to solve

$$\int \sin^m x \cos^n x dx.$$

- **If m is odd, then let $u = \cos x$.**
- **If n is odd, then let $u = \sin x$.**
- **If m and n are even, then use half-angle formula.**

Example 119 ***Evaluate***

$$\int \sin^2(x) \cos^3 x \, dx.$$

Example 120 ***Evaluate***

$$\int \sin^2(x)\, dx.$$

Example 121 ***Evaluate***

$$\int \sin^2(x) \cos^4 x\, dx.$$

7.3 Trigonometric Substitution

Trigonometric substitutions:

1. $\sqrt{a^2 - x^2}$**: Let** $x = a \sin\theta$**,** $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$**;**
2. $\sqrt{a^2 + x^2}$**: Let** $x = a \tan\theta$**,** $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$**;**
3. $\sqrt{x^2 - a^2}$**: Let** $x = a \sec\theta$**,** $0 \le \theta < \frac{\pi}{2}$ **or** $\pi \le \theta < \frac{3\pi}{2}$**.**

Example 122 ***Evaluate***

$$\int \sqrt{4 - x^2}\, dx.$$

Example 123 ***Evaluate***

$$\int_0^{1/\sqrt{2}} \frac{x^2}{\sqrt{1 - x^2}}\, dx.$$

Example 124 ***Evaluate***

$$\int \frac{1}{\sqrt{x^2 - 4}}\, dx$$

Example 125 ***Evaluate***

$$\int \frac{1}{\sqrt{4 + x^2}}\, dx$$

7.4 Integration of rational Functions by Partial Fractions

Consider a rational function

$$f(x) = \frac{P(x)}{Q(x)}.$$

By long division,

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},$$

where $P(x)$**,** $Q(x)$**,** $S(x)$ **and** $R(x)$ **are polynomials,** $\deg R < \deg Q$**.**

Partial fractions:

- **If** $Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$**, a product of distinct linea factors, then**

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}.$$

- **If some linear factors are same, for example,** $Q(x) = (a_1x+b_1)^n(a_2x+b_2)\cdots(a_kx+b_k)$**, then**

$$\frac{R(x)}{Q(x)} = \frac{A_{11}}{a_1x+b_1} + \frac{A_{12}}{(a_1x+b_1)^2} + \cdots + \frac{A_{1n}}{(a_1x+b_1)^n} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}.$$

- **If** $Q(x)$ **has an irreducible factor** ax^2+bx+c **without repeating, then**

$$\frac{R(x)}{Q(x)} = \frac{Ax+B}{ax^2+bx+c} + \cdots .$$

- **If** $Q(x)$ **has an irreducible factor** ax^2+bx+c **with repeating, e.g.,** $(ax^2+bx+c)^n$**, then**

$$\frac{R(x)}{Q(x)} = \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n} + \cdots .$$

Example 126 ***Evaluate***

$$\int \frac{4x+5}{(2x+3)(3x+4)}\,dx$$

Example 127 ***Evaluate***

$$\int \frac{x^3+2x^2-5x}{x^2+2x-8}\,dx$$

Example 128 ***Evaluate***

$$\int \frac{4x^2 + 8x}{(x+1)^2(x-1)}\, dx$$

Example 129 ***Evaluate***

$$\int \frac{3x^2 + 5x + 2}{x(x^2 + 2x + 2)}\, dx$$

7.5 Strategy for Integration

1. **Simplify the integrand if possible.**
2. **Look for an obvious substitution.**
3. **Classify the integrand according to its form.**

Example 130 ***Evaluate*** $\int \frac{1}{1-\sin x} dx$.

Example 131 ***Evaluate*** $\int \frac{1}{x(\ln x)^5} dx$.

7.7 Approximate Integration

Methods Approximating definite integral $\int_a^b f(x)dx$**: Let** $\Delta x = \frac{b-a}{n}$**,** $x_0 = a$**,** $x_1 = a + \Delta x$**,** $x_2 = a + 2\Delta x$**,...,** $x_n = a + n\Delta x = b$**.**

- **Left rule: uses the left endpoint of each subinterval.** L_n **denotes the result obtained by using "left rule" with** n **subintervals.**
- **Right rule: uses the right endpoint of each subinterval.** R_n **denotes the result obtained by using "right rule" with** n **subintervals.**
- **Midpoint rule: uses the midpoint of each subinterval.** M_n **denotes the result obtained by using "midpoint rule" with** n **subintervals.**
- **Trapezoid rule:**

$$T_n = \frac{L_n + R_n}{2} = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right].$$

- **Simpson's rule (using parabolas $y = ax^2 + bx + c$, n should be even):**

$$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right].$$

Remark. In Simpson's rule,

$$\int_{x_{i-1}}^{x_{i+1}} f(x)\,dx \approx \int_{x_{i-1}}^{x_i} (ax^2 + bx + c)\,dx = \frac{x_i - x_{i-1}}{3}[f(x_{i-1}) + 4f(x_i) + f(x_{i+1})].$$

Properties: 1) If f is increasing on $[a,b]$, then

$$L_n \le \int_a^b f(x)dx \le R_n.$$

2) If f is decreasing on $[a,b]$, then

$$R_n \le \int_a^b f(x)dx \le L_n.$$

3) If f is concave up on $[a,b]$, then

$$M_n \le \int_a^b f(x)dx \le T_n.$$

4) If f is concave down on $[a,b]$, then

$$T_n \le \int_a^b f(x)dx \le M_n.$$

5) $S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n$.

Example 132 ***Let*** $I = \int_0^2 \frac{1}{1+e^x}dx$.

1) Estimate I ***by using*** L_4, R_4, M_4, T_4, S_4.

2) Find the exact value of I.

3) Which is the best?

Theorem 9 ***(Error estimates:)***

$$\begin{aligned} E(M_n) &= \left|\int_a^b f(x)dx - M_n\right| \le \frac{(b-a)^3}{24n^2}\max_{x\in[a,b]}|f''(x)|, \\ E(T_n) &= \left|\int_a^b f(x)dx - T_n\right| \le \frac{(b-a)^3}{12n^2}\max_{x\in[a,b]}|f''(x)|, \\ E(S_n) &= \left|\int_a^b f(x)dx - S_n\right| \le \frac{(b-a)^5}{180n^4}\max_{x\in[a,b]}|f^{(4)}(x)|. \end{aligned}$$

Example 133 ***(1) Approximate***

$$\int_0^1 e^{-x^2}\,dx$$

by using T_4 and S_4, and estimate errors.

(2) How large should we take n in order to guarantee that the Simpson's Rule approximation for the integral is accurate to within 0.0001?

Remark. n **should be even.**

4.1 Maximum and Minimum Values

- **Absolute (Global) Maximum and Minimum:** $f(x)$ **has a Global (Absolute) Maximum at** p **if** $f(p) \geq f(x)$ **for all** x **in the domain;** $f(x)$ **has a Global (Absolute) Minimum at** p **if** $f(p) \leq f(x)$ **for all** x **in the domain;**

- **Local (or relative) extrema:** $f(x)$ **has a local minimum at** p **if** $f(p) \leq f(x)$ **for points** x **near** p**;** $f(x)$ **has a local maximum at** p **if** $f(p) \geq f(x)$ **for points** x **near** p**;**

- **Critical number: A point** p **in the domain such that** $f'(p) = 0$ **or** $f'(p)$ **undefined is called a critical number,** $(p, f(p))$ **is a critical point,** $f(p)$ **is a critical value.**

EXTREME VALUE THEOREM: If f(x) is continuous on a closed interval [a, b], then f attains an absolute maximum value f(c) and an absolute minimum value f(d) at some numbers c and d in [a, b].
FERMAT's THEOREM: If f has a local maximum or minimum at c, and if $f'(c)$ **exists, then** $f'(c) = 0$**.**
CLOSED INTERVAL METHOD: To find a global maximum or minimum for $f(x)$ **on a closed interval** $[a, b]$**:**

1. **Find all the critical numbers, e.g.,** x_1**, ...,** x_n**.**
2. **global minimum** $= \min\{f(x_1), ..., f(x_n), f(a), f(b)\}$**;**

   global maximum $= \max\{f(x_1), ..., f(x_n), f(a), f(b)\}$**.**

Example 134 ***Find the critical numbers of*** $f(x) = x^{3/5}(4 - x)$.

Sol: the critical numbers are 3/2 and 0.

Example 135 ***Find the global maximum and minimum of the function***

$$f(x) = 2x^3 - 3x^2 - 12x + 7, \quad [-2, 0].$$

Example 136 ***An object has the position function***

$$s(t) = e^{-t} \cos t.$$

Find the maximum distance to $s(0)$ ***for*** $t \geq 0$***.***

4.2 The Mean Value Theorem

MEAN VALUE THEOREM: Let f be a function that satisfies the following two hypotheses:

1. **f is continuous on the closed interval [a, b]**
2. **f is differentiable on the open interval (a, b)**

Then, there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Example 137 ***Suppose that f(0) = -3 and*** $f'(x) \leq 5$ ***for all values of x. How large can f(2) possibly be?***

4.3 and 4.5 How Derivatives Affect the Shapes of Curves

What Does f' **Say About** f**?**

Definition 20 $y = f(x)$ ***is increasing on an interval I if*** $f(x_1) \leq f(x_2)$ ***for any*** $x_1 < x_2, x_1, x_2 \in I$***;*** $y = f(x)$ ***is decreasing on an interval I if*** $f(x_1) \geq f(x_2)$ ***for any*** $x_1 < x_2, x_1, x_2 \in I$***.***

INCREASING/DECREASING TEST (I/D TEST):

- **If $f'(x) > 0$ on an interval, then f is increasing on that interval.**
- **If $f'(x) < 0$ on an interval, then f is decreasing on that interval.**
- **If $f'(x) = 0$ on an interval, then f is a constant on that interval.**

Example 138 ***Let*** $f(x) = x^4 - 4x^3 + 4x^2 + 4$***. State all the intervals of increase and decrease.***

First Derivative Test: Let p be a critical number. If f' changes from - to + at p, then f has a local minimum at p; If f' changes from + to - at p, then f has a local maximum at p.

Example 139 ***Let*** $f(x) = x^4 - 4x^3 + 4x^2 + 4$***.***
(a) Find all the critical numbers.
(b) State all the intervals of increase and decrease.
(c) Find all the local minimum points and all the local maximum points.

Example 140 ***Let*** $g(x) = x + 2\sin x$***,*** $0 \le x \le 2\pi$***.***
(a) Find all the critical numbers.
(b) State all the intervals of increase and decrease.
(c) Find all the local minimum points and all the local maximum points.

Definition 21 ***(CONCAVITY)If the graph of f lies above all of its tangents on an interval I (f' is increasing on I), it is called concave upward on I. If the graph of f lies below all of its tangents on I (f' is decreasing on I), it is called concave downward on I. If $f(x)$ changes concavity at p, then p is an inflection point, and $f''(p) = 0$ or undefined.***

CONCAVITY TEST: If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Second Derivative Test: Let p be a critical number. If $f''(p) > 0$, then f has a local minimum at p; If $f''(p) < 0$, then f has a local maximum at p; If $f''(p) = 0$, then nothing.

Example 141 ***Let*** $f(x) = x^4 - 4x^3 + 4x^2 + 4$***.***

(c) Find all the local minimum points and all the local maximum points by Second Derivative Test.

(d) Find all the points of inflection.

(e) State intervals of concavity.

(f) Sketch the graph.

Example 142 ***Consider the function***

$$f(x) = \frac{x}{x^2 - 1}.$$

Study the concavity and find all the points of inflection.

Example 143 ***Use the first and second derivatives of*** $f(x) = e^{1/x}$***, together with asymptotes, to sketch its graph.***

Example 144 ***Sketch the graph of*** f ***if*** f ***satisfies all of the following:***

(i) $f'(x) > 0$ ***on*** $(-\infty, 1)$***,*** $f'(x) < 0$ ***on*** $(1, \infty)$

(ii) $f''(x) > 0$ ***on*** $(-\infty, -2)$ ***and*** $(2, \infty)$***,*** $f''(x) < 0$ ***on*** $(-2, 2)$

(iii) $\lim_{x \to -\infty} f(x) = -2, \ \lim_{x \to \infty} f(x) = 0$***.***

4.4 Indeterminate Forms and L'Hospital's Rule

In this section, we are going to deal with the limit with the form:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 1^{\infty}, \quad 0 \cdot \infty, \quad 0^0, \ldots$$

L'Hospital's rule: If $\frac{f(x)}{g(x)}$ **becomes** $\frac{0}{0}$ **or** $\frac{\infty}{\infty}$ **as** $x \to x_0$**, where** x_0 **is finite or** ∞**, then**

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

Remark. $x \to x_0$ **can be replaced by any of the symbols** $x \to x_0^+$**,** $x \to x_0^-$**,** $x \to \infty$**, or** $x \to -\infty$**.**

Example 145 ***Calculate***

$$\lim_{t\to 0} \frac{\sin x}{x}, \quad \lim_{t\to 0} \frac{\sin x}{x^2}, \quad \lim_{t\to 0} \frac{e^t - t - 1}{t^2}.$$

Example 146 ***Calculate***

$$\lim_{x\to\infty} x^2 e^{-x}, \quad \lim_{x\to\infty} \left(1 - \frac{1}{x}\right)^x.$$

Example 147 ***Calculate***

$$\lim_{t\to 0} x^{\sin x}.$$

4.5 Summary of Curve Sketching

The following checklist is intended as a guide to sketching a curve y = f(x) by hand.

Not every item is relevant to every function. For instance, a given curve might not have an asymptote or possess symmetry. However, the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function.

- **A. DOMAIN**
- **B. INTERCEPTS**
- **C. SYMMETRY**
  1. **EVEN FUNCTION: f(-x) = f(x) for all x in D. the curve is symmetric about the y-axis. This means that our work is cut in half.**
  2. **ODD FUNCTION: f(-x) = -f(x) for all x in D. the curve is symmetric about the origin. This means that our work is cut in half.**
  3. **PERIODIC FUNCTION: f(x + p) = f(x) for all x in D, where p is a positive constant. The smallest such number p is called the period.**
- **D. ASYMPTOTES**
  - **HORIZONTAL:** $\lim_{x\to\pm\infty} f(x) = L$, **then** $y = L$ **is a HA.**
  - **VERTICAL:** $\lim_{x\to a^{\pm}} f(x) = \pm\infty$, **then** $x = a$ **is a VA.**
  - **SLANT: If** $\lim_{x\to\infty}[f(x) - (mx + b] = 0$, **y =mx + b is called a slant asymptote.**
- **E. INTERVALS OF INCREASE OR DECREASE: use I/D Test.**
- **F. LOCAL MAXIMUM AND MINIMUM VALUES: First Derivative Test or Second Derivative Test.**
- **G. CONCAVITY AND POINTS OF INFLECTION**

Example. Consider the function

$$f(x) = \left[x^2(6-x)\right]^{1/3}, \qquad -\infty < x < \infty.$$

We have (you don't need to check the following results!)

$$f'(x) = \frac{4-x}{\sqrt[3]{x(6-x)^2}}, \quad f''(x) = \frac{-8}{\sqrt[3]{x^4(6-x)^5}}.$$

a) Find all the critical numbers of $f(x)$.
Sol: From $f'(x) = 0$ we get $x = 4$;
From $f'(x)$ undefined we imply that $x = 0, 6$.
So we have three critical numbers $x = 4, 0, 6$.

b) Find all the intervals of increasing and decreasing, and classify all the critical numbers as local maxima, minima, or neither.
Sol: Look at the following table

x	$-\infty < x < 0$	$0 < x < 4$	$4 < x < 6$	$6 < x < \infty$
$f'(x)$	-	$+$	-	-
$f(x)$	**decreasing**	**increasing**	**decreasing**	**decreasing**

By the First Derivative Test, $f(0) = 0$ is a local minimum, $f(4) = \sqrt[3]{32}$ is a local maximum, $f(6)$ is neither.

c) Study the concavity and find the point(s) of inflection.
Sol: Note that $f''(x)$ undefined at $x = 0, 6$. Look at the following table

x	$-\infty < x < 0$	$0 < x < 6$	$6 < x < \infty$
$f''(x)$	-	-	$+$
$f(x)$	**concave down**	**concave down**	**concave up**

$x = 6$ or $(6, f(6))$ is a point of inflection.

d) Sketch the graph. (Remark. $f(x)$ has no vertical and horizontal asymptotes).

Example 148 ***Sketch the graph of:***

$$f(x) = \frac{2x^2}{x^2-4}, \quad f(x) = xe^x, \quad f(x) = \frac{\sin x}{2+\cos x}, \quad f(x) = \ln(1-x^2), \quad f(x) = \frac{2x^3}{x^2+1}.$$

4.7 Optimization Problems

GUIDELINES FOR SOLVING MAX./MIN. PROBLEMS

1. **If possible, draw a sketch or diagram of the problem to be solved. Pictures are a great help in organizing and sorting out your thoughts.**

2. **Define variables to be used and carefully label your picture or diagram with these variables. This step is very important because it leads directly or indirectly to the creation of mathematical equations.**

3. **Write down all equations which are related to your problem or diagram. Clearly denote that equation which you are asked to maximize or minimize. Experience will show you that MOST optimization problems will begin with two equations. One equation is a "constraint" equation and the other is the "optimization" equation. The "constraint" equation is used to solve for one of the variables. This is then substituted into the "optimization" equation before differentiation occurs. Some problems may have NO constraint equation. Some problems may have two or more constraint equations.**

4. **Change your equations to a function of only one variable. Then differentiate the function.**

5. **Verify that your result is a maximum or minimum value using the first or second derivative test for extrema.**

Example 149 ***Find two nonnegative numbers whose sum is 9 and so that the product of one number and the square of the other number is a maximum.***

Example 150 ***Build a rectangular garden with three parallel partitions using 500 feet of fencing. What dimensions will maximize the total area of the garden?***

Example 151 ***An open rectangular box with square base is to be made from surface area*** $48ft^2$***. What dimensions will maximize the volume of the box?***

Example 152 ***An open cylinder container has surface area $3\pi ft^2$. What dimensions will maximize the volume?***

Example 153 ***Find the point(s) (x, y) on the graph of $y = \sqrt{x}$ nearest the point $(4, 0)$.***

4.8 Newton's Method

Sometimes we are presented with a problem which cannot be solved by simple algebraic means. For instance, if we needed to find the roots of the polynomial

$$x^3 - x + 1 = 0,$$

we would find that the tried and true techniques just wouldn't work. However, we will see that calculus gives us a way of finding approximate solutions.
Newton's method: To approximate solutions of the equation $f(x) = 0$, start from x_1, we have approximate solutions

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

provided we have started with a good value for x_1 , this will produce approximate solutions to any degree of accuracy.
Example 154 ***Find the roots $\sqrt[6]{2}$ by Newton's method, correct to 8 decimals.***

Solution: Let $f(x) = x^6 - 2$. Then $\sqrt[6]{2}$ is a solution of the equation $f(x) = 0$. Note that $f(2) = 62 > 0$ and $f(0) = -2 < 0$. This tells us that the root is between 0 and 2. So we chose $x_1 = 1$ for our initial guess.

$$x_{n+1} = x_n - \frac{x_n^6 - 2}{6x_n^5} = \frac{5x_n^6 + 2}{6x_n^5}.$$

With our initial guess of $x_1 = 1$, we can produce the following values:

x_1	1
x_2	1.16666667
x_3	1.12644368
x_4	1.12249707
x_5	1.12246205
x_6	1.12246205

Notice how the values for x_n become closer and closer to the same value. This means that we have found the approximate solution to 8 decimal places.

Example 155 ***Find the roots of the polynomial*** $f(x) = x^3 - x + 1 = 0$ ***by Newton's method with*** $x_1 = -1$***, correct to 6 decimals.***

Solution: Note that $f(-2) = -5$ and $f(-1) = 1$. This tells us that the root is between -2 and -1. So we chose $x_1 = -1$ for our initial guess.

$$x_{n+1} = x_n - \frac{x_n^3 - x_n + 1}{3x_n^2 - 1} = \frac{2x_n^3 - 1}{3x_n^2 - 1}.$$

With our initial guess of $x_1 = -1$, we can produce the following values:

x_1	-1
x_2	-1.500000
x_3	-1.347826
x_4	-1.325200
x_5	-1.324718
x_6	-1.324717
x_7	-1.324717
x_8	-1.324717

Notice how the values for x_n become closer and closer to the same value. This means that we have found the approximate solution to six decimal places. In fact, this was obtained after only five relatively painless steps.