Ryerson University

Department of Mathematics

Test 3

MTH 240 Calculus II

Last Name:(**Print**)________________________. First Name (**Print**):________________________.

Student Number: ________________________. Signature: ________________________

Date: April 24, 2010, 9:00 am

Duration: 1hr. 30 min.

Professor (circle one)

Dr Homayouni Dr Olivares

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Section : ________

Instructions:

1. Have your student card available on your desk.
2. This is a closed-book test. **Notes, calculators and other aids are not permitted.** Verify that your test has pages 1-7.
3. Do not separate the pages of this test booklet.
4. The point value of each question is indicated by the question number.
5. Include all significant steps in your solutions to the questions, presented in the correct order. **Unjustified answers will be given little or no credit.** Cross out or erase all rough work not relevant to your solution.
6. Present your solutions neatly and legibly in the space provided. **Messy or illegible solutions will receive no credit.**
7. If you need more space, use the back of the previous page. Indicate this fact on the original page.

For Instructor's use only.

Question	Mark
1	
2	
3	
4	
5	
6	
Total	/50

1. [8 marks]

Find the radius of convergence and interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{2^n (x-3)^n}{\sqrt{n+3}}$$

Put $a_n = \dfrac{2^n (x-3)^n}{\sqrt{n+3}}$, then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1} (x-3)^{n+1}}{\sqrt{n+4}} \cdot \frac{\sqrt{n+3}}{2^n (x-3)^n} \right| = \lim_{n\to\infty} \left| 2(x-3) \frac{\sqrt{n+3}}{\sqrt{n+4}} \right| =$$

$$= 2|x-3| \lim_{n\to\infty} \frac{\sqrt{n+3}}{\sqrt{n+4}} = 2|x-3|$$

So, the series converges when $2|x-3| < 1 \Leftrightarrow$ $|x-3| < \frac{1}{2} \Leftrightarrow -\frac{1}{2} < x-3 < \frac{1}{2} \Leftrightarrow \frac{5}{2} < x < \frac{7}{2}$ and the radius of convergence is $R = \frac{1}{2}$.

when $x = \frac{5}{2}$ the series becomes $\sum_{n=1}^{\infty} \frac{2^n \cdot \frac{(-1)^n}{2^n}}{\sqrt{n+3}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$

$x = \frac{7}{2}$ the series becomes $\sum_{n=1}^{\infty} \frac{2^n \frac{1}{2^n}}{\sqrt{n+3}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}}$

The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$ converges by the A.S.T

The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}}$ diverges

So, the interval of convergence is $\left[\frac{5}{2}, \frac{7}{2} \right)$

2. [8 marks]

Find the Maclaurin series of $f(x) = \arctan \frac{1+2x}{1-2x}$ and its radius of convergence and interval of convergence.

$$f'(x) = \frac{1}{1+\left(\frac{1+2x}{1-2x}\right)^2} \cdot \left(\frac{1+2x}{1-2x}\right)' = \frac{(1-2x)^2}{(1-2x)^2+(1+2x)^2} \cdot \frac{2(1-2x)-(1+2x)\cdot(-2)}{(1-2x)^2} =$$

$$= \frac{2-4x+2+4x}{1-4x+4x^2+1+4x+4x^2} = \frac{4}{2+8x^2} = \frac{4}{2(1+4x^2)} = \frac{2}{1+4x^2}$$

$$= 2\cdot\frac{1}{1-(-4x^2)} = 2\cdot\sum_{n=0}^{\infty}(-4x^2)^n \quad \text{for } |-4x^2| = 4|x|^2 < 1 \Leftrightarrow |x| < \frac{1}{2}$$

$$= 2\sum_{n=0}^{\infty}(-4)^n \cdot x^{2n}$$

Now $f(x) = \int\left(2\cdot\sum_{n=0}^{\infty}(-4)^n \cdot x^{2n}\right)dx = 2\cdot\int\left(\sum_{n=0}^{\infty}(-4)^n x^{2n}\right)dx = 2\cdot\sum_{n=0}^{\infty}(-4)^n\int x^{2n}dx$

$$= 2\cdot\sum_{n=0}^{\infty}(-4)^n\cdot\frac{x^{2n+1}}{2n+1} + C \quad \text{for } |x|<\frac{1}{2} \text{ and } R=\frac{1}{2}$$

Since $f(0) = \arctan\frac{1+0}{1-0} = \arctan 1 = \frac{\pi}{4}$ we get $f(0) = 2\cdot\sum 0 + C$

$\Rightarrow C = \frac{\pi}{4}$.

So, if $x = -\frac{1}{2}$ $\quad 2\cdot\sum_{n=0}^{\infty}(-4)^n\cdot\frac{\frac{(-1)^{2n+1}}{2^{2n+1}}}{2n+1} = -2\sum_{n=0}^{\infty}\frac{(-1)^n}{2(2n+1)} = -\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}$

which converges by the A.S.T.

<u>REMARK</u>: $x = \frac{1}{2} \notin$ domain (f).

So, $\quad f(x) = 2\cdot\sum_{n=0}^{\infty}(-4)^n\cdot\frac{x^{2n+1}}{2n+1} + \frac{\pi}{4}$, $\quad -\frac{1}{2} \le x < \frac{1}{2}$ and $R = \frac{1}{2}$

$$= \sum_{n=0}^{\infty}(-1)^n\cdot\frac{2^{2n+1}}{2n+1}\cdot x^{2n+1} + \frac{\pi}{4}$$

3. [(5+3) marks] Given the function $f(x) = \sqrt[3]{1-x}$

(a) Find the Taylor polynomial $T_2(x)$ about the origin of the function $f(x)$.

$f(0) = 1$,

$f'(x) = \frac{1}{3}(1-x)^{-\frac{2}{3}}(-1) = -\frac{1}{3}(1-x)^{-\frac{2}{3}} \Rightarrow f'(0) = -\frac{1}{3}$

$f''(x) = \frac{2}{9}(1-x)^{-\frac{5}{3}} \cdot (-1) = -\frac{2}{9}(1-x)^{-\frac{5}{2}} \Rightarrow f''(0) = -\frac{2}{9}$

So, $T_2(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2$

$= 1 - \frac{1}{3}x - \frac{1}{9}x^2$

(b) Use the Taylor polynomial from (a) to approximate $\sqrt[3]{0.97}$

$f(x) \approx T_2(x)$

$\sqrt[3]{0.97} = \sqrt[3]{1-0.03} = \sqrt[3]{1-\frac{3}{100}} = f\left(\frac{3}{100}\right) \approx T_2\left(\frac{3}{100}\right) = 1 - \frac{1}{3}\cdot\frac{3}{100} - \frac{1}{9}\cdot\left(\frac{3}{100}\right)^2$

$= 1 - \frac{1}{100} - \frac{1}{10000} = \frac{10000-100-1}{10000} = \frac{9899}{10000} = 0.9899$

4. [(5+5) marks]

(a) If $u = x^4y + y^2z^3$, where

$$x = rse^t,\ y = rs^2e^{-t},\ z = r^2 s \sin t$$

find the value of $\frac{\partial u}{\partial s}$ when $r = 2,\ s = 1,\ t = 0$.

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$$= (4x^3y)\cdot re^t + (x^4 + 2yz^3)2rse^{-t} + 3y^2z^2(r^2\sin t)$$

when $r = 2$, $s = 1$, $t = 0$ we have $x = 2$, $y = 2$, $z = 0$

So, $\frac{\partial u}{\partial s}(2,1,0) = 64\cdot 2 + 16\cdot 4 + 0\cdot 0 = 192$

(b) If $z = f(x-y)$ where f is a differentiable function, show that $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$.

$\frac{\partial z}{\partial x} = f'(x-y)$, $\frac{\partial z}{\partial y} = f'(x-y)\cdot(-1)$

So, $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$

5. [(4 + 4) marks] Let $u = u(x, y)$ has continuous second-order partial derivatives and let

$$x = r - s \text{ and } y = r + 2s.$$

(a) Show that $\frac{\partial u}{\partial x} = \frac{2}{3}\frac{\partial u}{\partial r} - \frac{1}{3}\frac{\partial u}{\partial s}$. **[Hint:** Express r and s as functions of x and y]

Solving $x = r - s$, $y = r + 2s$ for r and s one gets

$r = \frac{2}{3}x + \frac{1}{3}y$, $s = -\frac{1}{3}x + \frac{1}{3}y$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} = \frac{2}{3}\frac{\partial u}{\partial r} - \frac{1}{3}\frac{\partial u}{\partial s}$$

$$\frac{\partial r}{\partial x} = \frac{2}{3}, \quad \frac{\partial s}{\partial x} = -\frac{1}{3}$$

$$\frac{\partial r}{\partial y} = \frac{1}{3}, \quad \frac{\partial s}{\partial y} = \frac{1}{3}$$

(b) Express $\frac{\partial^2 u}{\partial y \partial x}$ in terms of derivatives with respect to r and s.

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)\cdot\frac{\partial r}{\partial y} + \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right)\cdot\frac{\partial s}{\partial y}$$

$$= \frac{\partial}{\partial r}\left(\frac{2}{3}\frac{\partial u}{\partial r} - \frac{1}{3}\frac{\partial u}{\partial s}\right)\cdot\frac{\partial r}{\partial y} + \frac{\partial}{\partial s}\left(\frac{2}{3}\frac{\partial u}{\partial r} - \frac{1}{3}\frac{\partial u}{\partial s}\right)\cdot\frac{\partial s}{\partial y}$$

$$= \left(\frac{2}{3}\frac{\partial^2 u}{\partial r^2} - \frac{1}{3}\frac{\partial^2 u}{\partial r \partial s}\right)\cdot\frac{1}{3} + \left(\frac{2}{3}\frac{\partial^2 u}{\partial s \partial r} - \frac{1}{3}\frac{\partial^2 u}{\partial s^2}\right)\cdot\frac{1}{3}$$

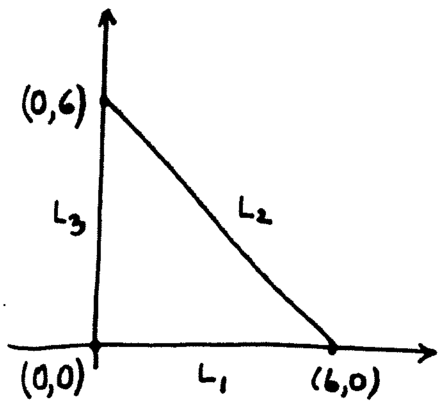
$$= \frac{2}{9}\frac{\partial^2 u}{\partial r^2} - \frac{1}{9}\frac{\partial^2 u}{\partial r \partial s} + \frac{2}{9}\frac{\partial^2 u}{\partial s \partial r} - \frac{1}{9}\frac{\partial^2 u}{\partial s^2}$$

$$= \frac{2}{9}\frac{\partial^2 u}{\partial r^2} + \frac{1}{9}\frac{\partial^2 u}{\partial r \partial s} - \frac{1}{9}\frac{\partial^2 u}{\partial s^2} \qquad \text{since } \frac{\partial^2 u}{\partial r \partial s} = \frac{\partial^2 u}{\partial s \partial r}$$

6. [8 marks] Find the absolute maximum and minimum values of

$$f(x,y) = 4xy^2 - x^2y^2 - xy^3$$

on the closed triangular region in the xy-plane with vertices $(0,0)$, $(0,6)$, $(6,0)$.



We first solve inside the region.

Here $\frac{\partial f}{\partial x} = 4y^2 - 2xy^2 - y^3$, $\frac{\partial f}{\partial y} = 8xy - 2x^2y - 3xy^2$

$\frac{\partial f}{\partial x} = 0 \Rightarrow y^2(4-2x-y) = 0 \Rightarrow y = 0$ OR $4 - 2x = y$.

But $y = 0$ isn't inside.

Substitute $y = 4 - 2x$ into $\frac{\partial f}{\partial y} = 0 \Leftrightarrow xy(8-2x-3y) =$

$\Rightarrow x = 0$ OR $4 - 2x = 0$ OR $8 - 2x - 12 + 6x = 0$

$x = 0$ OR $x = 2$ OR $x = 1$

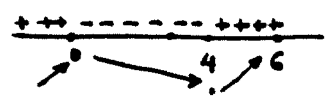
$x = 0$ isn't inside, when $x = 2$ we get $y = 0$ and $(2,0)$ isn't inside

Thus the only critical point inside is $(1,2)$ and $f(1,2) = 4$.

On L_1, $y = 0$ $f(x,0) = 0$.

On L_2, $x = -y + 6$ and $f(-y+6, y) = -2y^2(6-y) = -12y^2 + 2y^3$

$f' = -24y + 6y^2 = 6y(y-4)$



Critical points are $y = 0$, $y = 4$

$f(6,0) = 0$ and $f(2,4) = -64$

On L_3, $x = 0$, $f(0,y) = 0$.

So, on the whole region f attains the absolute max at $(1,2)$ with $f(1,2) = 4$ and the absolute min at $(2,4)$ with $f(2,4) = -64$.