Ryerson University
Department of Mathematics
MTH 240 Winter 2011 – Test II

LAST NAME: ______________________________ FIRST NAME: ______________________________
(Please print) (Please print)

I.D. NUMBER:________________________ SIGNATURE:________________________

Date: March 25, 2011 Duration: 90 minutes

Professor (circle one)

Kim	Poliakov
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Section:____

INSTRUCTIONS:

- Verify that your exam has 6 pages including this page.
- The use of notes, formula sheets, books or calculators is not allowed.
- For full-answer questions:

  Give full justification for your answers; correct answers alone may be worth nothing. Cross out or erase all rough work not relevant to your solution. Write your solutions in the space provided. If you need more space, use the back of the page. Indicate this fact on the original page, making sure that your solution cannot be confused with any rough work which may be there.

For markers' use only:

Page	Value	Mark
2	10	
3	12	
4	10	
5	10	
6	8	
Total	50	

1. (10 pts.) Solve the initial value problem:

$$y' - 3x^2 y = x^2, \quad y(0) = 0$$

Solution: This is a linear D.E. with $p(x) = -3x^2$ and $q(x) = x^2$.

The integrating factor $I(x)$ is

$$I(x) = e^{\int -3x^2 dx} = e^{-x^3}$$

Multiplying both sides of the D.E., we get

$$e^{-x^3} y' - 3x^2 e^{-x^3} y = x^2 e^{-x^3}$$

That is,

$$\left(e^{-x^3} y\right)' = x^2 e^{-x^3}$$

Integrating both sides of the above, we get

$$e^{-x^3} y = \int x^2 e^{-x^3} dx = -\frac{1}{3} e^{-x^3} + C$$

So, $y = Ce^{x^3} - \frac{1}{3}$
To determine C, we use the condition $y(0) = 0$. So,

$$0 = Ce^{0^3} - \frac{1}{3}$$

So, $C = \frac{1}{3}$,

and the solution to the above D.E. is $y = \frac{1}{3} e^{x^3} - \frac{1}{3}$

Way 2: As $y' = 3x^2y + x^2 = x^2(3y+1)$,

this is a separable D.E. Thus,

$$\frac{1}{3y+1}\,dy = x^2\,dx$$

Integrating both sides, we have

$$\int \frac{1}{3y+1}\,dy = \int x^2\,dx$$

$$\frac{1}{3}\ln|3y+1| = \frac{1}{3}x^3 + C'$$

$$|3y+1| = e^{3C'}\cdot e^{x^3}$$

$$3y+1 = Ce^{x^3}$$

i.e., $y = \frac{1}{3}(Ce^{x^3} - 1)$

To find C, we use the condition: $y(0) = 0$

$$0 = \frac{1}{3}(C-1)$$

So, $C = 1$

So, $y = \frac{1}{3}(e^{x^3} - 1)$

Way 3: Variation of Constants

Step 1. Find a general solution of the D.E.

$$y' - 3x^2 y = 0$$

It is separable, so

$$\frac{1}{y}\,dy = 3x^2\,dx$$

$$\int \frac{1}{y}\,dy = \int 3x^2\,dx$$

$$\ln|y| = x^3 + c'$$

$$y = ce^{x^3}$$

Step 2. We assume that the D.E. $y' - 3x^2 y = x^2$ has a solution of the form:

$$y = C(x)\,e^{x^3}$$

where $C(x)$ is an unknown function to be determined.

As $y' = C'(x)\,e^{x^3} + C(x)\cdot(e^{x^3}\cdot 3x^2)$,

$$[C'(x)\,e^{x^3} + 3x^2 C(x)\,e^{x^3}] - 3x^2(C(x)e^{x^3}) = x^2$$

$$C'(x)\,e^{x^3} = x^2$$

$$C'(x) = x^2 e^{-x^3}$$

So, $C(x) = \int x^2 e^{-x^3} dx$

$= -\frac{1}{3} e^{-x^3} + C$

So, we obtain a general solution of $y' - 3x^2 y = x^2$:

$$y = \left(-\frac{1}{3} e^{-x^3} + c\right) e^{x^3}$$

$$= c e^{x^3} - \frac{1}{3}$$

As $y(0) = 0$,

$$0 = c - \frac{1}{3}$$

So, $c = \frac{1}{3}$

So,

$$y = \frac{1}{3} e^{x^3} - \frac{1}{3}$$

2. (6+6 pts.) For each series below determine whether it is convergent. Justify your answer.

(a) $$\sum_{n=1}^{\infty} \frac{2^{n+1}}{e^n}$$

Solution: Let $a_n = \frac{2^{n+1}}{e^n}$. Note that

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+2}}{e^{n+1}} \div \frac{2^{n+1}}{e^n} = 2/e$$

So, this is a geometric series with $r = 2/e$.

As $r \in (-1, 1)$, this series is convergent.

(b) $$\sum_{n=1}^{\infty} \left(\sqrt{n^2+2n} - n\right)$$

Solution: Let $a_n = \sqrt{n^2+2n} - n$. Note that

$$\begin{aligned}
\lim_{n\to\infty} \left(\sqrt{n^2+2n} - n\right) &= \lim_{n\to\infty} \left(\left(\sqrt{n^2+2n} - n\right) \frac{\sqrt{n^2+2n}+n}{\sqrt{n^2+2n}+n}\right) \\
&= \lim_{n\to\infty} \frac{2n}{\sqrt{n^2+2n}+n} \\
&= \lim_{n\to\infty} \frac{2}{\sqrt{1+2/n}+1} \\
&= \cancel{1/2}\ 1 \\
&\neq 0
\end{aligned}$$

By the *Test for Divergence*, this series is divergent.

3. (10 pts.) Use the Integral Test or Comparison Test to determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(2e)^n}$ is convergent. *No credit will be given using other methods.*

Solution: **Method 1: Integral Test**

Let $a_n = \frac{n}{(2e)^n}$. We can take $f(x) = \frac{x}{(2e)^x}$. It is easy to see that $f(x)$ is continuous and positive on $[1, \infty)$. As

$$f'(x) = \frac{(2e)^x - x(2e)^x \ln(2e)}{(2e)^{2x}} < 0,$$

$f(x)$ is decreasing.

Now we examine whether the improper integral $\displaystyle\int_1^{\infty} \frac{x}{(2e)^x} dx$ is convergent. For that, we consider the limit $\displaystyle\lim_{t\to\infty} \int_1^t \frac{x}{(2e)^x} dx$. Let $u = x$ and $dv = \frac{1}{(2e)^x} dx$. Then $du = dx, v = -\frac{1}{(2e)^x \ln(2e)}$. By the integration by parts,

$$\begin{aligned}
\int_1^t \frac{x}{(2e)^x} dx &= \left(-\frac{x}{(2e)^x \ln(2e)}\right)_1^t + \int_1^t \frac{1}{(2e)^x \ln(2e)} dx \\
&= \left(\frac{1}{(2e)\ln(2e)} - \frac{t}{(2e)^t \ln(2e)}\right) + \left(-\frac{1}{(2e)^x \ln^2(2e)}\right)_1^t \\
&= \left(\frac{1}{(2e)\ln(2e)} - \frac{t}{(2e)^t \ln(2e)}\right) + \left(\frac{1}{(2e)\ln^2(2e)} - \frac{1}{(2e)^t \ln^2(2e)}\right)
\end{aligned}$$

As $\displaystyle\lim_{t\to\infty} \frac{t}{(2e)^t \ln(2e)} = 0$ and $\displaystyle\lim_{t\to\infty} \frac{1}{(2e)^t \ln^2(2e)} = 0$,

$$\lim_{t\to\infty} \int_1^t \frac{x}{(2e)^x} dx = \frac{1}{(2e)\ln(2e)} + \frac{1}{(2e)\ln^2(2e)}.$$

So, the improper integral $\displaystyle\int_1^{\infty} \frac{x}{(2e)^x} dx$ is convergent. By the Integral Test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(2e)^n}$ is convergent.

Method 2: Comparison Test

As $n \le e^n$, we have $a_n = \frac{n}{(2e)^n} \le \frac{1}{2^n}$.

As the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $r = 1/2 \in (-1, 1)$. So, it is convergent.

By the Comparison Test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(2e)^n}$ is convergent.

4. (10 pts.) Does the series converge absolutely, conditionally, or diverge? Justify your answer.

$$\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{\sqrt{n+5}}$$

Solution: This is an alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$, where $b_n = \frac{1}{\sqrt{n+5}}$.

Note that $\lim_{n\to\infty} \frac{1}{\sqrt{n+5}} = 0$, and $b_n = \frac{1}{\sqrt{n+5}} > \frac{1}{\sqrt{n+6}} = b_{n+1}$ for all $n \geq 1$, so $\{b_n\}$ is decreasing.

By the Alternating Series Test, we conclude that the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n+5}}$ is convergent.

Consider the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+5}}$. As

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+5}} \div \frac{1}{\sqrt{n}} = 1,$$

and the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p = 1/2 < 1$, so is divergent.

By the Limit Comparison Test, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+5}}$ is divergent.

From the above analysis, we conclude that the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n+5}}$ is ***convergent conditionally***.

5. (8 pts.) Determine whether the series is convergent or divergent. Justify your answer.

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{n^{2n}}$$

Solution: Let $a_n = \frac{(n!)^2}{n^{2n}}$. Note that

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{((n+1)!)^2}{(n+1)^{2(n+1)}}}{\frac{(n!)^2}{n^{2n}}} \\
&= \lim_{n\to\infty} \frac{n^{2n}}{(n+1)^{2n}} \\
&= \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^{2n}} \\
&= \frac{1}{\left(\lim_{n\to\infty}(1+\frac{1}{n})^n\right)^2} \\
&= 1/e^2 \\
&< 1
\end{aligned}$$

By the Ratio Test, we conclude that the series is convergent.

Way 2: By the Comparison Test.

As $\frac{n!}{n^n} \leq \frac{1}{n}$ for all $n \geq 1$,

$a_n = \frac{(n!)^2}{n^{2n}} = \left(\frac{n!}{n^n}\right)^2 \leq \frac{1}{n^2}$ for all $n \geq 1$

As $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$,

it is convergent.

By the Comparison Test, $\sum_{n=1}^{\infty} \frac{(n!)^2}{n^{2n}}$ is convergent.