Ryerson University
Department of Mathematics
MTH 240 Winter 2011 – Test I

LAST NAME: ______________________________ FIRST NAME: ______________________________
(Please print) (Please print)

I.D. NUMBER: ________________________ SIGNATURE: ________________________

Date: February 18, 2011 Duration: 90 minutes

Professor (circle one)

Kim	Poliakov
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Section: _____

<u>INSTRUCTIONS</u>:

- Verify that your exam has **SIX (6)** pages including this page.
- The use of notes, formula sheets, books or calculators is not allowed.
- **The tests written in pencil or erasable pens are not eligible for remarking.**
- For full-answer questions:

  Give full justification for your answers; correct answers alone may be worth nothing. Cross out or erase all rough work not relevant to your solution. Write your solutions in the space provided. If you need more space, use the back of the page. Indicate this fact on the original page, making sure that your solution cannot be confused with any rough work which may be there.

For markers' use only:

Page	Value	Mark
2	10	
3	10	
4	10	
5	10	
6	10	
Total	50	

1. (5+5 pts.) Evaluate the following indefinite integrals.

(a) $$\int \sin(\sqrt{x})\, dx$$

Soln: Make a substitution: $y = \sqrt{x}$. So, $dy = \frac{1}{2\sqrt{x}}dx$.

$$\int \sin(\sqrt{x})\, dx = \int 2y \sin y\, dy$$

Let $u = 2y, dv = \sin y dy$. Then $du = 2dy, v = -\cos y$

$$\begin{aligned} \int 2y \sin y\, dy &= (2y)(-\cos y) - \int (-\cos y)(2dy) \\ &= -2y\cos y + 2\sin y + C \end{aligned}$$

So,

$$\int \sin(\sqrt{x})\, dx = -2\sqrt{x}\cos(\sqrt{x}) + 2\sin(\sqrt{x}) + C$$

(b) $$\int (\sec x)^4\, dx$$

Soln:

$$\begin{aligned} \int (\sec x)^4\, dx &= \int \left(\tan^2 x + 1\right)\sec^2 x dx \\ &= \int (\tan^2 x)(\sec^2 x)dx + \int \sec^2 x dx \\ &= \int (\tan^2 x)(\sec^2 x)dx + \tan x \end{aligned}$$

For $\int (\tan^2 x)(\sec^2 x)dx$, we let $u = \tan x$. So $du = \sec^2 x dx$, and

$$\begin{aligned} \int (\tan^2 x)(\sec^2 x)dx &= \int u^2 du \\ &= \frac{1}{3}u^3 + C \\ &= \frac{1}{3}\tan^3 x + C \end{aligned}$$

So,

$$\int (\sec x)^4\, dx = \frac{1}{3}\tan^3 x + \tan x + C$$

2. (5+5 pts.) Evaluate the following limits.

(a) $$\lim_{x\to 0^+} x^x$$

Soln:

$$\begin{aligned}\lim_{x\to 0^+} x^x &= \lim_{x\to 0^+} e^{\ln x^x} \\ &= \lim_{x\to 0^+} e^{x\ln x} \\ &= e^{\lim_{x\to 0^+} x\ln x}\end{aligned}$$

For $\lim_{x\to 0^+} x\ln x$, we apply L'Hospital's Rule as follows:

$$\begin{aligned}\lim_{x\to 0^+} x\ln x &= \lim_{x\to 0^+} \frac{\ln x}{1/x} \\ &= \lim_{x\to 0^+} \frac{1/x}{-1/x^2} \\ &= \lim_{x\to 0^+} -x \\ &= 0\end{aligned}$$

So,

$$\lim_{x\to 0^+} x^x = e^0 = 1$$

(b) $$\lim_{x\to 0} \frac{\int_{\cos x}^{1} \sin(\sqrt{t})\,dt}{x\tan x}$$

Soln: It is $\frac{0}{0}$ type. We apply L'Hospital's Rule as follows:

$$\begin{aligned}\lim_{x\to 0} \frac{\int_{\cos x}^{1} \sin(\sqrt{t})\,dt}{x\tan x} &= \lim_{x\to 0} \frac{\sin(\sqrt{\cos x})\sin x}{\tan x + x\sec^2 x} \\ &= \lim_{x\to 0} \frac{\sin(\sqrt{\cos x})}{\sec x + \frac{x}{\sin x}\sec^2 x} \\ &= \frac{\sin(1)}{2}\end{aligned}$$

3. (6+4 pts.) Determine if the improper integrals below are convergent or divergent. If divergent, justify your answer; if convergent, evaluate it.

(a) $$\int_1^\infty \frac{(\ln x)^2}{x^2}\,dx$$

Soln: Let $u = (\ln x)^2$ and $dv = \frac{1}{x^2}dx$. So, $du = 2\ln x \cdot \frac{1}{x}dx$ and $v = -\frac{1}{x}$.

$$\begin{aligned}\int_1^t \frac{(\ln x)^2}{x^2}\,dx &= \left((\ln x)^2\left(-\frac{1}{x}\right)\right)_1^t - \int_1^t \left(-\frac{1}{x}\right)\left(2\ln x\cdot\frac{1}{x}dx\right)\\ &= -\frac{\ln^2 t}{t} + \int_1^t \frac{2\ln x}{x^2}dx\end{aligned}$$

To evaluate $\int_1^t \frac{2\ln x}{x^2}\,dx$, we let $u = 2\ln x$ and $dv = \frac{1}{x^2}dx$. So, $du = \frac{2}{x}dx$ and $v = -\frac{1}{x}$.

$$\begin{aligned}\int_1^t \frac{2\ln x}{x^2}\,dx &= \left(2\ln x\left(-\frac{1}{x}\right)\right)_1^t - \int_1^t \left(-\frac{1}{x}\right)\left(\frac{2}{x}dx\right)\\ &= -\frac{2\ln t}{t} + \int_1^t \frac{2}{x^2}dx\\ &= -\frac{2\ln t}{t} + \left(-\frac{2}{x}\right)_1^t = -\frac{2\ln t}{t} - \frac{2}{t} + 2\end{aligned}$$

So, $\int_1^t \frac{(\ln x)^2}{x^2}\,dx = -\frac{\ln^2 t}{t} - \frac{2\ln t}{t} - \frac{2}{t} + 2$

Using L'Hospital's Rule, we have $\lim_{t\to\infty} \frac{\ln^2 t}{t} = 0$ and $\lim_{t\to\infty} \frac{2\ln t}{t} = 0$.

$$\lim_{t\to\infty}\int_1^t \frac{(\ln x)^2}{x^2}\,dx = \lim_{t\to\infty}\left(-\frac{\ln^2 t}{t} - \frac{2\ln t}{t} - \frac{2}{t} + 2\right) = 2$$

(b) $$\int_0^{1/2} \frac{47}{\sqrt{1-4x^2}}\,dx$$

Soln: Let $y = 2x$. Then $dy = 3dx$, and

$$\begin{aligned}\int \frac{47}{\sqrt{1-4x^2}}\,dx &= \int \frac{47}{\sqrt{1-y^2}}\left(\frac{1}{2}dy\right)\\ &= \frac{47}{2}\arcsin y + C = \frac{47}{2}\arcsin(2x) + C\end{aligned}$$

$$\begin{aligned}\lim_{t\to\frac{1}{2}^-}\int_0^t \frac{47}{\sqrt{1-4x^2}}dx &= \lim_{t\to\frac{1}{2}^-}\left(\frac{47}{2}\arcsin(2x)\right)_0^t\\ &= \lim_{t\to\frac{1}{2}^-}\frac{47}{2}\arcsin(2t) = \frac{47\pi}{4}\end{aligned}$$

So, $\int_0^{1/2} \frac{47}{\sqrt{1-4x^2}}\,dx = \lim_{t\to\frac{1}{2}^-}\int_0^t \frac{47}{\sqrt{1-4x^2}}dx = \frac{47\pi}{4}$.

4. (10 pts.) Evaluate the following integral (Hint: $\int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C$).

$$\int_1^2 \frac{dx}{\sqrt{x^2 - 2x + 4}}.$$

Soln: Make a substitution: $y = x - 1$. Then $dy = dx$, and $y : 0 \to 1$.

$$\int_1^2 \frac{dx}{\sqrt{x^2 - 2x + 4}} = \int_0^1 \frac{1}{\sqrt{y^2 + 3}} dy$$

Make a trigonometric substitution: $y = \sqrt{3}\tan\theta$. Then

$$\sqrt{y^2 + 3} = \sqrt{3}\sec\theta$$

and

$$dy = \sqrt{3}\sec^2\theta d\theta.$$

$$\begin{aligned}
\int_0^1 \frac{1}{\sqrt{y^2 + 3}} dy &= \int_0^{\pi/6} \left(\frac{1}{\sqrt{3}\sec\theta}\right) \sqrt{3}\sec^2\theta d\theta \\
&= \int_0^{\pi/6} \sec\theta d\theta \\
&= (\ln|\sec\theta + \tan\theta|)\,|_0^{\pi/6} \\
&= \ln(\sqrt{3})
\end{aligned}$$

5. (10 pts.) Evaluate the following integral.

$$\int \frac{x^3+x+1}{x^4+4x^2}dx$$

Soln: As $x^4+x^2 = x^2(x^2+4)$,

$$\frac{x^3+x+1}{x^4+4x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4}$$

So,

$$\begin{aligned} x^3+x+1 &= Ax(x^2+4) + B(x^2+4) + (Cx+D)x^2 \\ &= (A+C)x^3 + (B+D)x^2 + 4Ax + 4B \end{aligned}$$

So,

$$\begin{aligned} A+C &= 1 \\ B+D &= 0 \\ 4A &= 1 \\ 4B &= 1 \end{aligned}$$

So,

$$\begin{aligned} A &= 1/4 \\ B &= 1/4 \\ C &= 3/4 \\ D &= -1/4 \end{aligned}$$

$$\begin{aligned} \int \frac{x^3+x+1}{x^4+x^2}dx &= \int \left(\frac{\frac{1}{4}}{x} + \frac{\frac{1}{4}}{x^2} + \frac{\frac{3}{4}x - \frac{1}{4}}{x^2+4} \right) dx \\ &= \int \left(\frac{\frac{1}{4}}{x} + \frac{\frac{1}{4}}{x^2} \right) dx + \int \frac{\frac{3}{4}x}{x^2+4}dx - \int \frac{\frac{1}{4}}{x^2+4}dx \\ &= \frac{1}{4}\ln|x| - \frac{1}{4}\cdot\frac{1}{x} + \frac{3}{8}\ln(x^2+4) - \frac{1}{8}\arctan\left(\frac{x}{2}\right) + C \end{aligned}$$