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Applied Ordinary Differential Equations
ENGR 213 - Section F
Prof. Alina Stancu

Exam II (A)

(1) (6 points) Solve the homogeneous ODE

$$x^2y'' + 5xy' + 4y = 0.$$

Solution: The characteristic equation of this homogeneous Cauchy-Euler ODE is $m(m-1) + 5m + 4 = 0$ or $m^2 + 4m + 4 = 0$. It has a double root $r = -2$. Hence the general solution of the ODE on $x > 0$ (or $x < 0$) is

$$y(x) = c_1x^{-2} + c_2x^{-2}\ln x, \quad c_{1,2} = \text{constants}.$$

(2) (14 points) Solve the initial value problem

$$y'' + 4y' + 5y = e^{-2x}, \quad y(0) = 0,\ y'(0) = 1.$$

Solution: Consider first the associated homogeneous ODE: $y'' + 4y' + 5y = 0$ with the characteristic equation $r^2 + 4r + 5 = 0$ whose roots are $r_{1,2} = -2 \pm i$. Hence

$$y_c(x) = c_1e^{-2x}\cos x + c_2e^{-2x}\sin x, \quad c_{1,2} = \text{constants}.$$

We now look for a particular solution y_p to the non-homogenous ODE. We'll use here the method of undetermined coefficients by setting $y_p(x) = Ae^{-2x}$. As $y_p(x) = -2Ae^{-2x}$ and $y_p''(x) = 4Ae^{-2x}$, we deduce that $4Ae^{-2x} - 8Ae^{-2x} + 5Ae^{-2x} = e^{-2x} \quad \Rightarrow \; A = 1$ and $y_p(x) = e^{-2x}$.

Thus

$$y_{general}(x) = c_1e^{-2x}\cos x + c_2e^{-2x}\sin x + e^{-2x}, \quad c_{1,2} = \text{constants}.$$

We'll now use the initial conditions to find $c_{1,2}$. As $y(0) = 0$, we have $c_1 + 1 = 0 \;\Rightarrow\; c_1 = -1$. Evaluating $y'(x) = c_1(-2e^{-2x}\cos x - e^{-2x}\sin x) + c_2(-2e^{-2x}\sin x + e^{-2x}\cos x) - 2e^{-2x}$, thus $y'(0) = -2c_1 + c_2 - 2 = c_2 = 1$.

Therefore the solution of the IVP is

$$y(x) = -e^{-2x}\cos x + e^{-2x}\sin x + e^{-2x}.$$

(3) (10 points) Use the variation of parameters to solve the differential equation

$$y'' + y = \cos^2 x.$$

Solution: The complementary part of the solution follows from $r^2+1 = 0 \Rightarrow r = \pm i$ and is

$$y_c(x) = c_1 \cos x + c_2 \sin x.$$

Considering $y_1(x) = \cos x,\ y_2(x) = \sin x$, the Wronskian is $W(x) = 1 \neq 0$ for all real x's. To find the complementary solution we calculate $W_1(x) = \det \begin{pmatrix} 0 & \sin x \\ \cos^2 x & \cos x \end{pmatrix} = -\sin x \cos^2 x$ and $W_2(x) = \det \begin{pmatrix} \cos x & 0 \\ -\sin x & \cos^2 x \end{pmatrix} = \cos^3 x$.

The method of variation of parameters gives $y_p(x) = y_1(x)u_1(x) + y_2(x)u_2(x)$, where $u_1'(x) = W_1(x)/W(x)$ and $u_2'(x) = W_2(x)/W(x)$.

Integrating (by taking $u = \cos x \quad du = -\sin x\, dx$) and taking the constant of integration to be zero, we have

$$u_1(x) = \int (-\sin x \cos^2 x)\, dx = \int u^2\, du = \frac{u^3}{3} = \frac{\cos^3 x}{3}.$$

On the other hand,

$$u_2(x) = \int \cos^3 x\, dx = \int \cos x\,(1 - \sin^2 x)\, dx = \int (1 - u^2)\, du = u - \frac{u^3}{3} = \sin x - \frac{\sin^3 x}{3},$$

where above we used the fundamental identity of trigonometry ($\sin^2 x + \cos^2 x = 1$) and the substitution $u = \sin x,\ du = \cos x\, dx$.

Consequently,

$$y_p(x) = \cos x \cdot \frac{\cos^3 x}{3} + \sin x \cdot \left(\sin x - \frac{\sin^3 x}{3}\right)$$

and

$$y(x) = c_1 \cos x + c_2 \sin x + \frac{\cos^4 x}{3} + \sin^2 x - \frac{\sin^4 x}{3}, \quad c_{1,2} = \text{arbitrary constants.}$$

(4) (10 points) A mass weighing 10 pounds stretches a spring 0.5 foot. Determine the equation of motion if the mass is initially released from a point 6 inches below the equilibrium position with a downward velocity of 4 ft/s. What is the instantaneous velocity at the first time when the mass passes through the equilibrium position?

Solution: The equation of motion is $mx'' + kx = 0$, where $m = 10/32$ slug and $k = 10/0.5 = 20$ ft/lb. Thus

$$x'' + 64x = 0 \quad \Rightarrow \quad x(t) = c_1 \cos 8t + c_2 \sin 8t,\ c_{1,2} = \text{constants.}$$

To determine the constants, use the initial conditions. As $x(0) = 1/2$ ft, $c_1 = 1/2$. Additionally, $x'(t) = -8c_1 \sin 8t + 8c_2 \cos 8t$, thus $x'(0) = 4$ ft/sec and $8c_2 = 4$.

Consequently, $x(t) = \frac{1}{2}\cos 8t + \frac{1}{2}\sin 8t$ and $x'(t) = -4\sin 8t + 4\cos 8t$. To find the time when the mass passes through the equilibrium position, set $x(t) = 0$. Note that this implies $\tan(8t) = -1$ whose first positive solution is for $8t = 3\pi/4$. So

$$x'(3\pi/32) = -4\,\frac{\sqrt{2}}{2} - 4\,\frac{\sqrt{2}}{2} = -4\sqrt{2}\ \text{ft/sec}$$

is the answer needed.