tutorial section

COMP1805-BCM Discrete Structures I

Assignment 4

Due Date: March 24, 2016

Time Due: Before 3:00pm in the drop box in HP 3115

Grade / 34

Student Number ____________________ **Student Name** ____

- Write down your name and student number on **every** page.
- You must have a cover page that clearly states **your name, student number, course number and tutorial section**. If you do not have a cover page with this information, your assignment **will not be marked.**
- The questions should be answered in order and your assignment sheets must be stapled, otherwise the assignment will not be marked.
- Every part of every question is worth 2 marks. The grading scheme for each question is 2 for correct answer, 1 for an answer that is not completely correct and 0 otherwise.

1. Determine which of the following simple graphs is feasible. If the graph is feasible draw an example and if not explain what is the problem.

   (a) G has five vertices with the following degrees $\deg(v_1) = 1, \deg(v_2) = 5, \deg(v_3) = 2, \deg(v_4) = 4, \deg(v_5) = 2$.

   Solution: The graph is not a simple graph because $\deg(v_2) = 5$. The graph has only five vertices and therefore, the maximum degree of a vertex is 4.

   (b) G has six vertices with the following degrees $\deg(v_1) = 2, \deg(v_2) = 4, \deg(v_3) = 4, \deg(v_4) = 4, \deg(v_5) = 2, \deg(v_6) = 2$.

   Solution: Yes the graph is feasible.

   (c) G has six vertices with the following degrees $\deg(v_1) = 2, \deg(v_2) = 1, \deg(v_3) = 1, \deg(v_4) = 1, \deg(v_5) = 2, \deg(v_6) = 2$.

   Solution: the graph is not feasible because it has an odd number of vertices each with an odd degree.

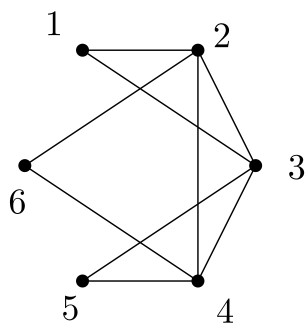
   (d) G has five vertices with the following degrees $\deg(v_1) = 4, \deg(v_2) = 1, \deg(v_3) = 1, \deg(v_4) = 1, \deg(v_5) = 1$.

   Solution: Yes the graph is feasible ($K_{1,4}$).

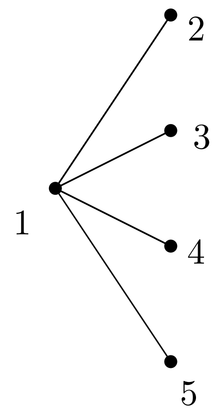
   (e) G has five vertices with the following degrees $\deg(v_1) = 3, \deg(v_2) = 3, \deg(v_3) = 2, \deg(v_4) = 2, \deg(v_5) = 2$.

   Solution: Yes the graph is feasible ($K_{2,3}$).

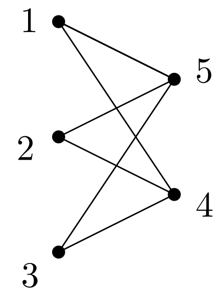
   The feasible graphs are drawn in the figure below:



(b)



(d)



(e)

2. Draw the following graphs

   (a) Undirected graph represented by the adjacency matrix

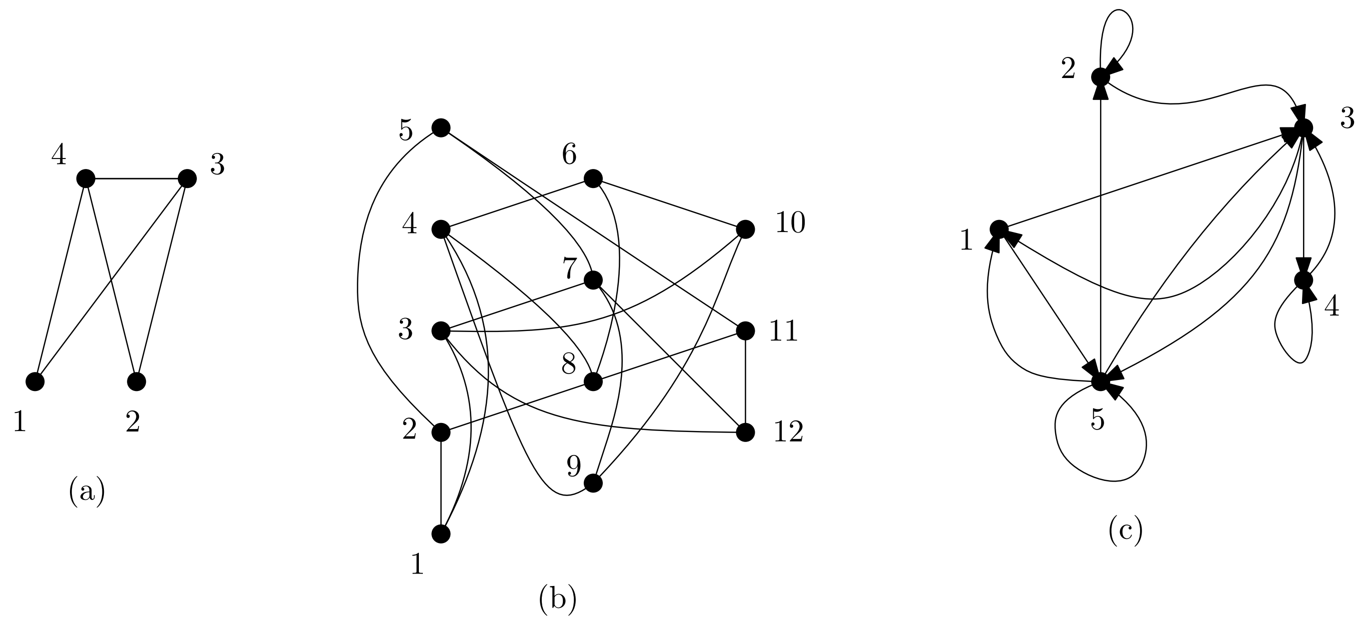
$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

   (b) Undirected graph represented by adjacency list.

1:	2, 3, 4
2:	1, 5, 8
3:	1, 7, 10, 12
4:	1, 6, 8, 9
5:	2, 7, 11
6:	4, 8, 10
7:	3, 5, 9, 12
8:	2, 4, 6, 11
9:	4, 7, 10
10:	3, 6, 9
11:	5, 8, 12
12:	3, 7, 11

   (c) Directed graph represented by the adjacency matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$



3. Given the adjacency matrix M in Question 2c, compute M^2. What do the numbers in the resulting matrix represent? Look at the corresponding graph of the M that you drew in Question 2c.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 2 \\ 1 & 0 & 1 & 2 & 1 \\ 2 & 2 & 3 & 1 & 3 \end{pmatrix}$$

Each entry $m(i, j)$ in the M^2 represents the number of different paths that exist from v_i to v_j of length at most 2.

4. Definition: A graph $G = (V, E)$ can be coloured with k colours if $\forall v \in V$, v can be coloured with one of k colours and $\forall e = \{u, v\} \in E$, u, v are coloured with different colours. A graph $G = (V, E)$ is bipartite if the vertex set V can be partitioned into 2 sets $A, B \subset V$, such that $A \cap B = \emptyset$ and $A \cup B = V$, and $\forall e = \{u, v\} \in E$, we have that $u \in A$ and $v \in B$.

Prove that a graph $G = (V, E)$ is bipartite if and only if G can be coloured with 2 colours.

Proof. In order to prove that $G = (V, E)$ is bipartite if and only if G can be coloured with 2 colours, we need to show that $G = (V, E)$ is bipartite $\to$ G can be coloured with 2 colours; and G can be coloured with 2 colours $\to$ $G = (V, E)$ is bipartite. Let the two colours be red and blue.

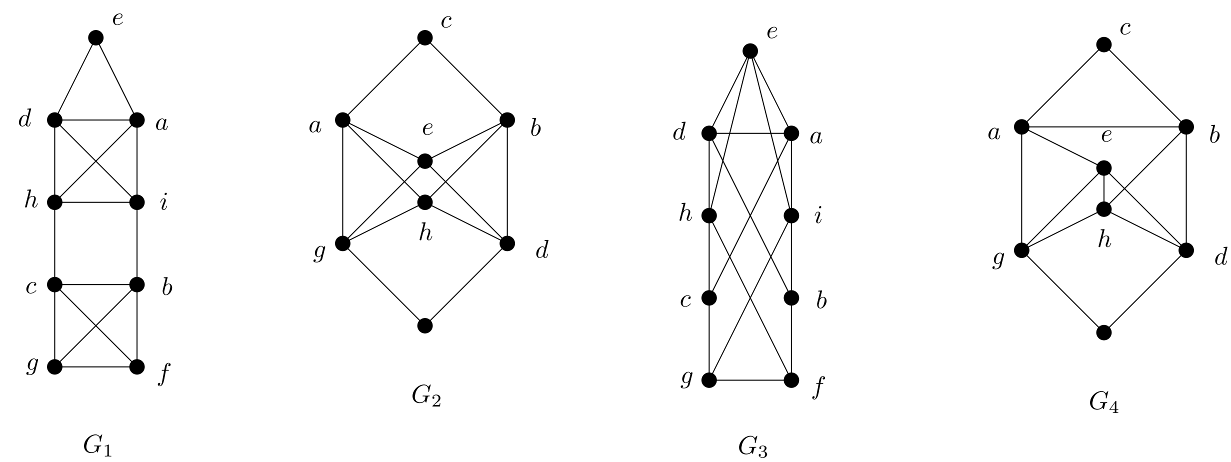
- We first prove that if a graph is bipartite then it can be coloured with two colours. If $G = (V, E)$ is bipartite, this means that V can be partitioned into two sets A, B such that all the edges in the graph have one endpoint in A and the other in B. Colour the vertices in A red and the B blue. This is a 2-colouring since every edge has endpoints with different colours.
- We now show that if G has a 2-colouring then G is bipartite. If G has a 2-colouring, this means that the vertices can be assigned one of 2 colours such that every edge has different coloured endpoints. Let A be the vertices of one colour and B the vertices of the other colour. Since every vertex is assigned one of two colours, $A \cup B = V$ and $A \cap B = \emptyset$. Since no edge has both endpoints in A or both in B, we conclude that the graph is bipartite.

∎

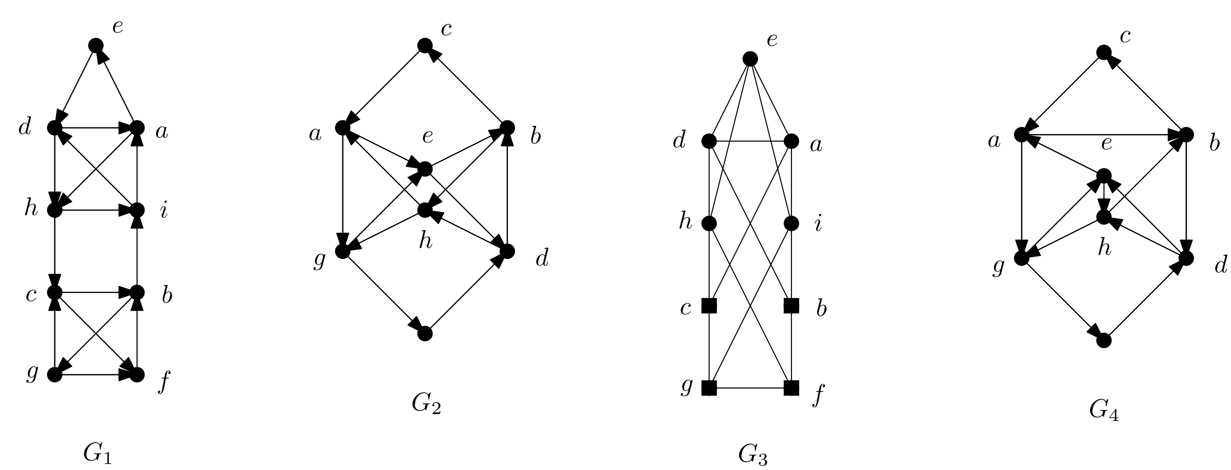
5. Given a graph G with $n \geq 4$ vertices, where n is even, prove that if every vertex has degree $n/2+1$, then G must contain a clique of size 3. Recall that a clique of size 3 is cycle on 3 vertices. Therefore, you must show that there always exists 3 vertices a, b, c in G such that ab is an edge, bc is an edge and ac is an edge.

   Proof. Let $e = v_1v_2$ be an edge of G. If both v_1 and v_2 are adjacent to a vertex v_i, then we have a 3-cycle. Since v_1 has degree $n/2+1$ it must be adjacent to $n/2$ vertices other than v_2. Similarly v_2 must be adjacent to $n/2$ vertices other than v_1. If v_1 and v_2 are not adjacent to a common vertex, then they must be adjacent to $2 * n/2 = n$ other vertices. However, there are only $n-2$ vertices in $V \setminus \{v_1, v_2\}$. Therefore, v_1 and v_2 must be adjacent to a common vertex giving a 3-cycle. ∎

6. Determine which of the simple graphs in the figure below has an Euler cycle or Euler Path (Euler Tour). If there is an Euler path or cycle, then show it on the graph. If no Euler cycle or tour exists, explain why none exists.



Graph G_1 has an Euler path starting at g and ending at f. Graphs G_2 and G_4 have Euler Tours. Graph G_3 does not have an Euler Tour or Path because it has 4 vertices of odd degree which are highlighted with squares in the figure. The paths and tours are drawn directly on the graphs in the figure below:



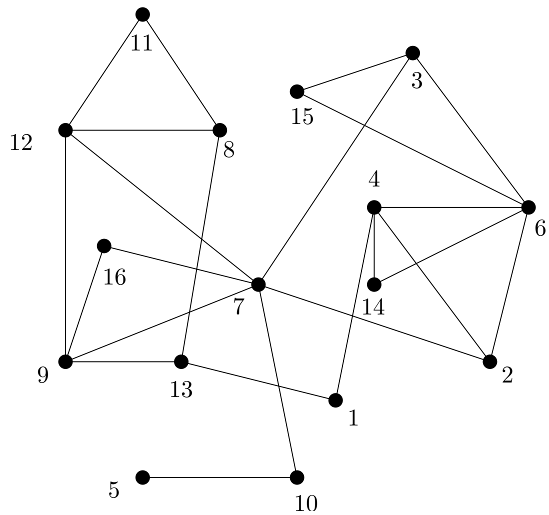
7. Given the graph $G = (V, E)$ in the figure below, compute:

   (a) the BFS tree,

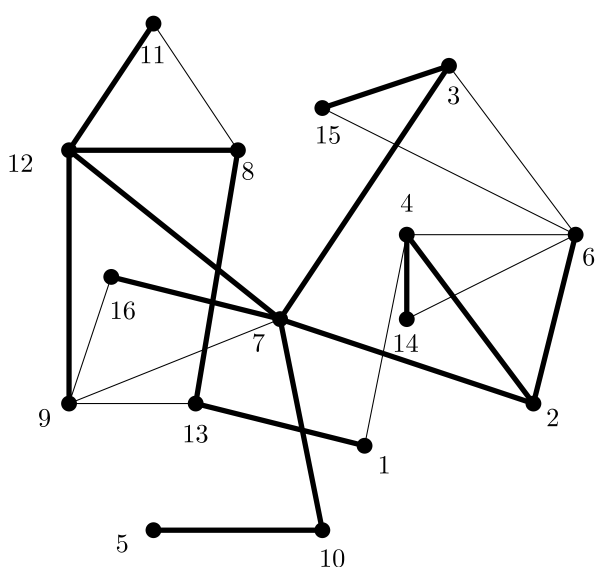
(b) the DFS tree.

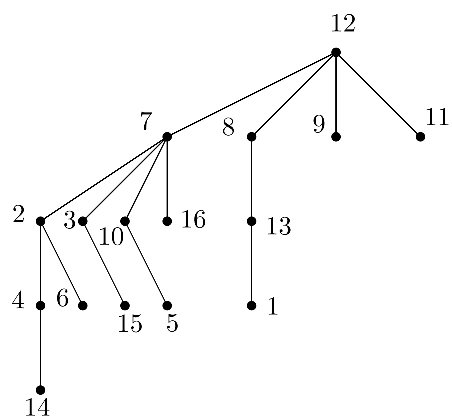
In doing so start with vertex 12 for both trees. For BFS, if a vertex has several adjacent vertices then process the vertices in sorted order from smallest to largest index. For example, vertex 12 has four adjacent vertices $11, 8, 7, 9$. Process vertex 7 first, then 8, 9 and finally, 11. Similarly, for DFS, if a vertex has several possible options, select the one with smallest index.

To output the tree, you may draw it on the graph itself, or provide the adjacency list of the tree where the vertices in each list are sorted from smallest to largest.

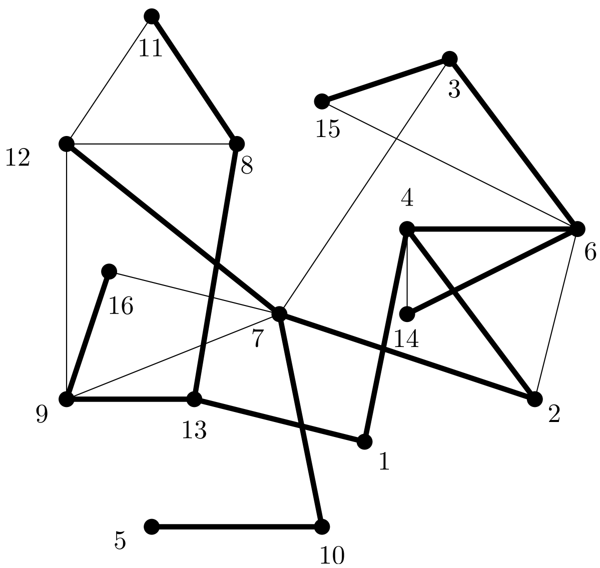


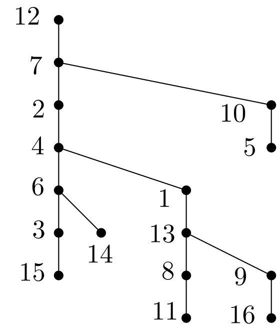
Solution given in the figures below:





Resulting BFS tree





Resulting DFS Tree