MAT2322C - Second Midterm Solutions

1. $\vec{r}(t) = (2+3t^2)\hat{\imath} + (4+t^2)\hat{\jmath} + (1-t^2)\hat{k},\ t \geq 0.$

a) When the object reaches the given position we have:

$$(2+3t^2)\hat{\imath} + (4+t^2)\hat{\jmath} + (1-t^2)\hat{k} = \frac{11}{4}\hat{\imath} + \frac{17}{4}\hat{\jmath} + \frac{3}{4}\hat{k}$$

So, $(2+3t^2) = \frac{11}{4}$, $(4+t^2) = \frac{17}{4}$, $(1-t^2) = \frac{3}{4}$.

Choosing any of these relationships:

$$(2+3t^2) = \frac{11}{4} \Rightarrow 3t^2 = \frac{11}{4} - 2 \Rightarrow t^2 = \frac{1}{3}\,\frac{3}{4} = \frac{1}{4}$$

As $t \geq 0$, $t = \frac{1}{2}$ //.

b) $\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = 6t\,\hat{\imath} + 2t\,\hat{\jmath} - 2t\,\hat{k}$ //

$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = 6\,\hat{\imath} + 2\,\hat{\jmath} - 2\,\hat{k}$ //

2. $\vec{F}(x,y,z) = x\,\hat{i} + 2zy\,\hat{j} + x\,\hat{k}$.

a) Along C_1: $\vec{r}(t) = t\,\hat{i} + t^2\,\hat{j} + t^3\,\hat{k}$ for $0 \le t \le 1$.

So, along this path $x(t) = t$, $y(t) = t^2$, $z(t) = t^3$

Now we use $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$.

$\vec{r}\,'(t) = \hat{i} + 2t\,\hat{j} + 3t^2\,\hat{k}$

$\vec{F}(\vec{r}(t)) = t\,\hat{i} + 2t^5\,\hat{j} + t\,\hat{k}$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \left(t\,\hat{i} + 2t^5\,\hat{j} + t\,\hat{k}\right) \cdot \left(\hat{i} + 2t\,\hat{j} + 3t^2\,\hat{k}\right) dt$$

$$= \int_0^1 \left(t + 4t^6 + 3t^3\right) dt$$

$$= \left(\frac{t^2}{2} + 4\frac{t^7}{7} + 3\frac{t^4}{4}\right)\Bigg|_0^1 = \frac{1}{2} + \frac{4}{7} + \frac{3}{4} = \frac{28+32+42}{56}$$

$$= \frac{102}{56} = \frac{51}{28}$$

b) The straight line goes from vector $\vec{r}_0 = 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k}$ to the vector $\vec{r}_1 = 1\,\hat{i} + 1\,\hat{j} + 1\,\hat{k}$. We find a parametric representation for the line by writing:

$$\vec{r}(t) = (\vec{r}_1 - \vec{r}_0)\,t + \vec{r}_0 \quad , \quad 0 \le t \le 1.$$

$\vec{r}(t) = (\hat{i} + \hat{j} + \hat{k})t = t\hat{i} + t\hat{j} + t\hat{k}$

So: $\vec{r}'(t) = \hat{i} + \hat{j} + \hat{k}$

$\vec{F}(\vec{r}(t)) = t\hat{i} + 2t^2\hat{j} + t\hat{k}$.

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt = \int_0^1 (t\hat{i} + 2t^2\hat{j} + t\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k})\, dt$$

$$= \int_0^1 (t + 2t^2 + t)\, dt = 2\int_0^1 (t^2 + t)\, dt = 2\left(\frac{t^3}{3} + \frac{t^2}{2}\right)\Big|_0^1$$

$$= 2\left(\frac{1}{3} + \frac{1}{2}\right) = 2\,\frac{5}{6} = \frac{5}{3}$$

3. a) $\vec{F}(x,y) = x^2\,\hat{i} + y^2\,\hat{j}$

If $\vec{F} = \nabla f$ then $\frac{\partial f}{\partial x} = x^2$ (1), $\frac{\partial f}{\partial y} = y^2$ (2)

So: $f(x,y) = \frac{x^3}{3} + g(y)$.

Now we perform the derivative of this function with respect to y and use (2):

$f_y = \frac{\partial g}{\partial y} = y^2 \Rightarrow g(y) = \frac{y^3}{3} + K$, where K is a constant.

Thus: $f(x,y) = \frac{x^3}{3} + \frac{y^3}{3} + k$ //

$\vec{F} = \nabla f$, $\vec{F}$ is conservative //

b) $\vec{F}(x,y) = 2xy\,\hat{i} + x^2\,\hat{j}$

If $\vec{F} = \nabla f$ then $\frac{\partial f}{\partial x} = 2xy$, $\frac{\partial f}{\partial y} = x^2$

So: $f(x,y) = x^2 y + g(y)$

Now $\frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y} = x^2 \Rightarrow \frac{\partial g}{\partial y} = 0$

$g(y) = k$, where k is a constant.

Then: $f(x,y) = x^2 y + k$. //

$\vec{F} = \nabla f$, $\vec{F}$ is conservative. //

c) $\vec{F}(x,y) = 2xy\,\hat{i} + 3x^2\,\hat{j}$.

If $\vec{F} = \nabla f$ then $\frac{\partial f}{\partial x} = 2xy$, $\frac{\partial f}{\partial y} = 3x^2$

So: $f(x,y) = x^2 y + g(y)$

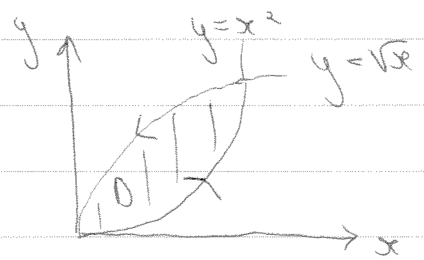
Now: $\frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y} = 3x^2 \Rightarrow \frac{\partial g}{\partial y} = 2x^2$

But $g = g(y)$ and hence its derivative can not depend on x.
This means that we can not find f such that $\vec{F} = \nabla f$.

$\vec{F}$ is not conservative //

4. $\vec{F}(x,y) = (2x^2 + 3y)\,\hat{i} + (2x + 3y^2)\,\hat{j}$

The curve C is sketched below:



Using Green's Theorem:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \left[(2x^2 + 3y)\,\hat{i} + (2x + 3y^2)\,\hat{j} \right] \cdot \left[dx\,\hat{i} + dy\,\hat{j} \right]$$

$$= \int_C (2x^2 + 3y)\,dx + (2x + 3y^2)\,dy$$

$$= \iint_D \left[\frac{\partial}{\partial x}(2x + 3y^2) - \frac{\partial}{\partial y}(2x^2 + 3y) \right] dA$$

$$= \iint_D (2-3)\, dA = -\iint_D dA$$

On D: y goes from x^2 to $\sqrt{x}$

x goes from 0 to 1.

So: $$\int_C \vec{F}\cdot d\vec{r} = -\int_0^1\int_{x^2}^{\sqrt{x}} dy\, dx = -\int_0^1 y\Big|_{x^2}^{\sqrt{x}} dx$$

$$= -\int_0^1 \left(\sqrt{x} - x^2\right) dx = -\left(\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right)\Bigg|_0^1$$

$$= -\left(\frac{2}{3} - \frac{1}{3}\right) = -\frac{1}{3}$$

5. $\vec{F}(x,y,z) = e^y\,\hat{\imath} + \cos(x)\,\hat{\jmath} + (z^2 - y)\,\hat{k}$

Curl of $\vec{F}$: $$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^y & \cos x & z^2 - y \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(z^2 - y) - \cancelto{0}{\frac{\partial}{\partial z}\cos x}\right)\hat{\imath} - \left(\cancelto{0}{\frac{\partial}{\partial x}(z^2-y)} - \cancelto{0}{\frac{\partial}{\partial z}e^y}\right)\hat{\jmath}$$

$$+ \left(\frac{\partial}{\partial x}\cos x - \frac{\partial}{\partial y}e^y\right)\hat{k} = -\hat{\imath} - (\sin x + e^y)\,\hat{k}$$

Divergence of $\vec{F}$:

$$\nabla \cdot \vec{F} = \cancelto{0}{\frac{\partial}{\partial x} e^{y}} + \cancelto{0}{\frac{\partial}{\partial y} \cos(x)} + \frac{\partial}{\partial z}(z^2 - y)$$

$$= 2z$$