MAT 2322 C - First Midterm solutions

1. $f(x,y) = 8y^3 + 12x^2 - 24xy$

Critical points: $f_x = 24x - 24y = 0 \Rightarrow y = x$ (1)
$f_y = 24y^2 - 24x = 0 \Rightarrow y^2 = x$ (2)

From (1) and (2); $y^2 = y \Rightarrow y=0$ or $y=1$

As $y=x$, the critical points are $(0,0)$ and $(1,1)$.

Classification: $D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - [f_{xy}(x,y)]^2$
$= 24 \cdot 48 \cdot y - 24^2$

At $(0,0)$, $D(0,0) = -24^2 < 0$. So, $(0,0)$ is a saddle point.

At $(1,1)$, $D(1,1) = 24 \cdot 48 - 24^2 > 0$.
$f_{xx}(1,1) = 24 > 0$.
$(1,1)$ is a local minimum.

2. $f(x,y) = 3x + y$, $g(x,y) = x^2 + y^2 = 10$.

At the absolute extrema $\nabla f = \lambda \nabla g$.

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So, $f_x = \lambda g_x \Rightarrow 3 = 2\lambda x \Rightarrow x = 3/2\lambda$

$f_y = \lambda g_y \Rightarrow 1 = 2\lambda y \Rightarrow y = 1/2\lambda$

$g = 10 \Rightarrow x^2+y^2 = 10$ ③

Substituting ① and ② in ③:

$$\frac{9}{4\lambda^2} + \frac{1}{4\lambda^2} = 10 \Rightarrow \lambda^2 = \frac{1}{4} \Rightarrow \lambda = \pm\frac{1}{2}$$

i) For $\lambda = \frac{1}{2}$, $x = 3$, $y = 1$

$f(3,1) = 10$.

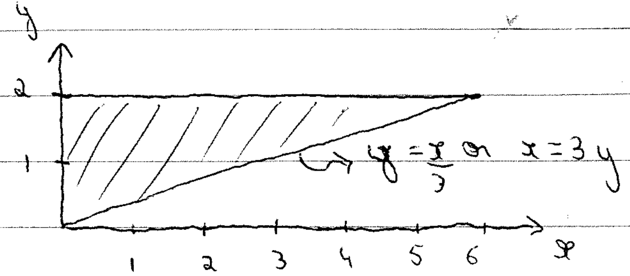
ii) For $\lambda = -\frac{1}{2}$, $x = -3$, $y = -1$

$f(-3,-1) = -10$.

The absolute maximum of the function on the circle is $f(3,1) = 10$ and the absolute minimum is $f(-3,-1)$

3. $$\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1}\, dy\, dx$$

The region of integration is:



So, x goes from 0 to $3y$
y goes from 0 to 2

Inverting the integral:

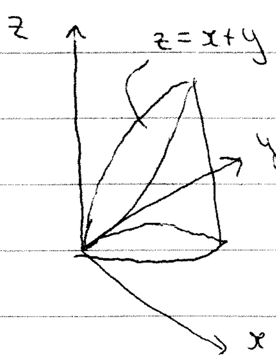
$$\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1}\,dy\,dx = \int_0^2 \int_0^{3y} x\sqrt{y^3+1}\,dx\,dy$$

$$= \int_0^2 \sqrt{y^3+1}\left[\frac{x^2}{2}\Big|_0^{3y}\right]dy = \frac{9}{2}\int_0^2 y^2\sqrt{y^3+1}\,dy$$

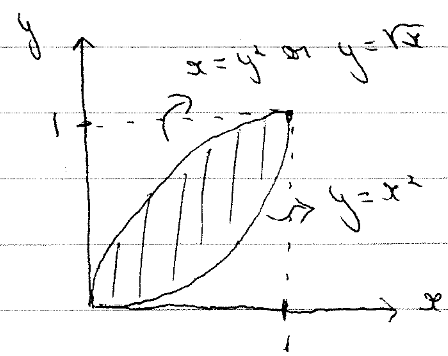
$$= \frac{9}{2}\left[\frac{(y^3+1)^{3/2}}{3/2}\,\frac{1}{3}\Big|_0^2\right] = 9^{3/2} - 1 = 26$$

4. $V = \iiint_E dV$

The solid is:



On x-y plane:



So, z goes from 0 to $x+y$
y goes from x^2 to $\sqrt{x}$.
x goes from 0 to 1.

$$V = \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y} dz\, dy\, dx = \int_0^1 \int_{x^2}^{\sqrt{x}} (x+y)\, dy\, dx$$

$$= \int_0^1 \left(xy + \frac{y^2}{2} \Big|_{x^2}^{\sqrt{x}} \right) dx = \int_0^1 \left(x^{3/2} + \frac{x}{2} - x^3 - \right.$$

$$= \left(\frac{x^{5/2}}{5/2} + \frac{x^2}{4} - \frac{x^4}{4} - \frac{x^5}{10} \Big|_0^1 \right)$$

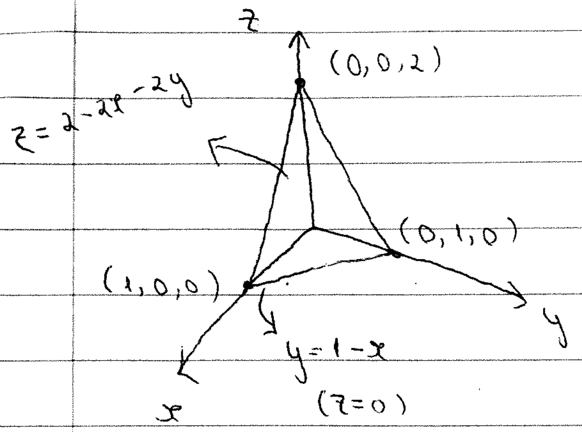
$$= \frac{2}{5} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{4}} - \frac{1}{10} = \frac{3}{10} //$$

5. $M = \iiint_E \rho(x,y,z)\, dV$

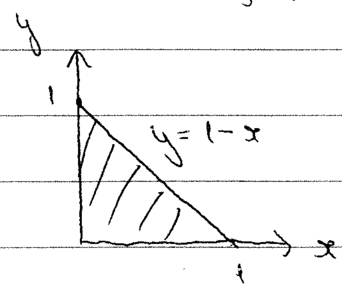
The general equation of a plane is $ax + by + cz = d$.
From the points $(1,0,0)$, $(0,1,0)$ and $(0,0,2)$, we

$a = d$, $y = d$, $c = d/2$, then $x + y + \frac{z}{2} = 1$

The tetrahedron is:



On x-y plane:



So, z goes from 0 to $2-2x-2y$

y goes from 0 to $1-x$

x goes from 0 to 1.

$$M = \int_0^1 \int_0^{1-x} \int_0^{2-2x-2y} y \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} y(2-2x-2y) \, dy \, dx$$

$$= \int_0^1 \left(y^2 - xy^2 - \frac{2}{3} y^3 \Big|_0^{1-x} \right) dx$$

$$= \int_0^1 \left[(1-x)(1-x)^2 - \frac{2}{3}(1-x)^3 \right] dx = \int_0^1 \frac{(1-x)^3}{3} \, dx$$

$$= \left[-\frac{(1-x)^4}{12} \right]_0^1 = \frac{1}{12}$$