MAT 2322 Assignment # 8: Solutions

1. unit sphere $\vec{r}(\phi,\theta) = \sin\phi\cos\theta\,\hat{\imath} + \sin\phi\sin\theta\,\hat{\jmath} + \cos\phi\,\hat{k}$

we know that $|\vec{r}_\phi \times \vec{r}_\theta| = \sin\phi$ (last assignment)

so $\displaystyle\iint_S f\,dS = \iint_D f(\vec{r}(\phi,\theta))\,|\vec{r}_\phi \times \vec{r}_\theta|\,d\phi\,d\theta$

$$= \int_0^{2\pi}\int_0^{\pi} (\sin\phi\cos\theta)^2 \sin\phi \; d\phi\, d\theta$$

$$= \int_0^{2\pi} \cos^2\theta\, d\theta \int_0^{\pi} \sin^3\phi\, d\phi$$

$$= \int_0^{2\pi} \frac{1}{2}(1+\cos(2\theta))\,d\theta \int_0^{\pi} (1-\cos^2\phi)\sin\phi\, d\phi$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\Big|_0^{2\pi}\right)\left(\frac{1}{3}\cos^3\phi - \cos\phi\Big|_0^{\pi}\right)$$

$$= \frac{1}{2}\left[(2\pi + 0) - 0\right]\left[\left(\frac{1}{3}(-1)+1\right) - \left(\frac{1}{3} - 1\right)\right]$$

$$= (\pi)(4/3)$$

$$= \boxed{4\pi/3}$$

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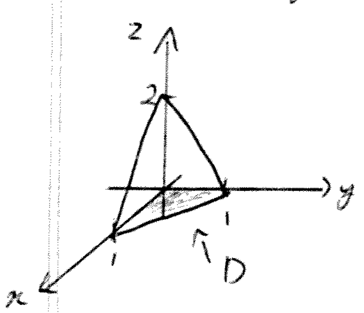
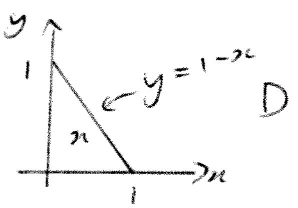
2. the surface is $z = g(x,y) = 2 - 2x - 2y$ (ie a graph)





so $\displaystyle\iint_S f\,dS = \iint_D f(x,y,g(x,y))\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\; dx\,dy$

$$= \int_0^1 \int_0^{1-y} (x + y + 2 - 2x - 2y)\sqrt{1+(-2)^2+(-2)^2}\,dx\,dy$$

$$= \int_0^1 \int_0^{1-y} (2 - x - y)(3)\,dx\,dy$$

$$= 3\int_0^1 \left(2x - \tfrac{1}{2}x^2 - xy \Big|_0^{1-y} \right) dy$$

$$= 3\int_0^1 \left(2(1-y) - \tfrac{1}{2}(1-y)^2 - y(1-y)\right) dy$$

$$= 3\int_0^1 \left(2 - 2y - \tfrac{1}{2}(1 - 2y + y^2) - y + y^2\right) dy$$

$$= 3\int_0^1 \left(\tfrac{3}{2} - 2y + \tfrac{1}{2}y^2\right) dy$$

$$= 3\left(\tfrac{3}{2}y - y^2 + \tfrac{1}{6}y^3 \Big|_0^1\right)$$

$$= 3\left(\tfrac{3}{2} - 1 + \tfrac{1}{6} - 0\right) = \boxed{2}$$

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3. sphere of radius a : $\vec{r}(\phi,\theta) = a\sin\phi\cos\theta\,\hat{i} + a\sin\phi\sin\theta\,\hat{j} + a\cos\phi\,\hat{k}$

$\vec{r}_\phi \times \vec{r}_\theta = a^2\sin^2\phi\cos\theta\,\hat{i} + a^2\sin^2\phi\sin\theta\,\hat{j} + a^2\sin\phi\cos\phi\,\hat{k}$

(which is outward) (↑ last assignment)

so $\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(\phi,\theta))\cdot \vec{r}_\phi\times\vec{r}_\theta\,d\phi\,d\theta$

$$= \int_0^{\pi/2}\int_0^{\pi/2} (a\cos\phi\,\hat{k})\cdot\left(a^2\sin^2\phi\cos\theta\,\hat{i} + a^2\sin^2\phi\sin\theta\,\hat{j} + a^2\sin\phi\cos\phi\,\hat{k}\right) d\phi\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} a^3\sin\phi\cos^2\phi\,d\phi\,d\theta$$

$$= \left(\pi/2\right)\left(a^3\right)\left(-\tfrac{1}{3}\cos^3\phi \Big|_0^{\pi/2}\right) = \boxed{\pi a^3/6}$$

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4. the cone is $z = \sqrt{x^2+y^2} = r$

and so $\vec{r}(\theta, z) = z\cos\theta\, \hat{\imath} + z\sin\theta\, \hat{\jmath} + z\hat{k}$, $0 \le z \le 1$

(or $\vec{r}(r,\theta) = r\cos\theta\, \hat{\imath} + r\sin\theta\, \hat{\jmath} + r\hat{k}$)

$\vec{r}_\theta = -z\sin\theta\, \hat{\imath} + z\cos\theta\, \hat{\jmath}$

$\vec{r}_z = \cos\theta\, \hat{\imath} + \sin\theta\, \hat{\jmath} + \hat{k}$

then $\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -z\sin\theta & z\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix}$

$= z\cos\theta\, \hat{\imath} + z\sin\theta\, \hat{\jmath} - z\hat{k}$ (which is outward)

so $\displaystyle\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(\theta,z))\cdot \vec{r}_\theta \times \vec{r}_z \, d\theta\, dz$

$\displaystyle = \int_0^1 \int_0^{2\pi} \left((z\cos\theta)(z\sin\theta)\hat{\imath} - z\hat{k}\right)\cdot\left(z\cos\theta\,\hat{\imath} + z\sin\theta\,\hat{\jmath} - z\hat{k}\right) d\theta\, dz$

$\displaystyle = \int_0^1 \int_0^{2\pi} \left(z^2\cos^2\theta\sin\theta + z^2\right) d\theta\, dz$

$\displaystyle = \int_0^1 z^2\, dz \int_0^{2\pi} \left(\cos^2\theta\sin\theta + 1\right) d\theta$

$$= \left(1/3\right)\left(-\tfrac{1}{3}\cos^3\theta + \theta \Big|_0^{2\pi}\right) = \boxed{2\pi/3}$$

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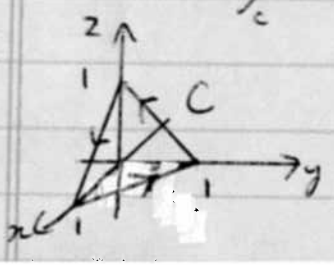
5. want $\oint_C \vec{F}\cdot d\vec{r}$, use Stokes' thm means calculate $\iint_S \nabla\times\vec{F}\cdot d\vec{S}$



S is graph $z = 1 - x - y$

C ccw $\Rightarrow$ S is upward

so $\vec{n} = -\frac{\partial g}{\partial x}\hat{\imath} - \frac{\partial g}{\partial y}\hat{\jmath} + \hat{k} = \hat{\imath} + \hat{\jmath} + \hat{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & xz & x^2 \end{vmatrix} = -x\hat{\imath} - 2x\hat{\jmath} + (z-1)\hat{k}$$

so $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot d\vec{S} = \iint_D \nabla\times\vec{F}\cdot\vec{n}\,dx\,dy$

$= \int_0^1\int_0^{1-y} (-x\hat{\imath} - 2x\hat{\jmath} + (-x-y)\hat{k})\cdot(\hat{\imath}+\hat{\jmath}+\hat{k})\,dx\,dy$

$= \int_0^1\int_0^{1-y} (-4x - y)\,dx\,dy$

$= \int_0^1 \left(-2x^2 - xy \Big|_0^{1-y}\right)dy$

$= \int_0^1 \left(-2(1-y)^2 - y(1-y)\right)dy$

$= \int_0^1 \left(-2(1-2y+y^2) - y + y^2\right)dy$

$= \int_0^1 (-2 + 3y - y^2)\,dy = \left(-2y + \frac{3}{2}y^2 - \frac{1}{3}y^3 \Big|_0^1\right) = \boxed{-5/6}$

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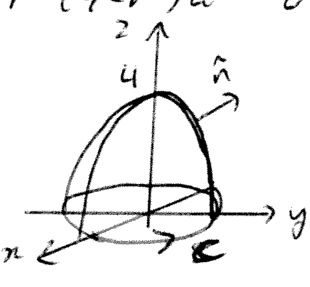
6. want $\iint_S \nabla\times\vec{F}\cdot d\vec{S}$, use Stokes' Thm means calculate $\oint_C \vec{F}\cdot d\vec{r}$

S is $\vec{r}(r,\theta) = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath} + (4-r^2)\hat{k} \quad 0\le r\le 2,\ 0\le\theta\le 2\pi$

$z = 4 - r^2 \Rightarrow$ paraboloid



S upward $\Rightarrow$ C ccw

C is circle of radius 2 : $\vec{r}(t) = 2\cos t\,\hat{\imath} + 2\sin t\,\hat{\jmath}$ $\quad 0 \le t \le 2\pi$

so $\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r} = \int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$

$= \int_0^{2\pi} (0\,\hat{\imath} + 6\cos t\,\hat{\jmath} + 10\sin t\,\hat{k}) \cdot (-2\sin t\,\hat{\imath} + 2\cos t\,\hat{\jmath})\,dt$

$= \int_0^{2\pi} 12\cos^2 t\,dt$

$= \int_0^{2\pi} 6(1 + \cos(2t))\,dt$

$= 6\left(t + \frac{1}{2}\sin(2t)\,\Big|_0^{2\pi}\right) = \boxed{12\pi}$