MAT 2371: SOLUTIONS TO MIDTERM

Short Answer

Question 1. For $i \in \{1,2,3,4\}$, let A_i be the event that component i is working. Let W be the event that the circuit is working. Then

$$\begin{aligned}
\mathbb{P}[W] &= \mathbb{P}[A_1 \cap ((A_2 \cap A_3) \cup A_4)] \\
&= \mathbb{P}[A_1 \cap A_2 \cap A_3] + \mathbb{P}[A_1 \cap A_4] - \mathbb{P}[A_1 \cap A_2 \cap A_3 \cap A_4] \\
&= \mathbb{P}[A_1]\mathbb{P}[A_2]\mathbb{P}[A_3] + \mathbb{P}[A_1]\mathbb{P}[A_4] - \mathbb{P}[A_1]\mathbb{P}[A_2]\mathbb{P}[A_3]\mathbb{P}[A_4] \\
&= (0.7)(0.8)(0.6) + (0.7)(0.5) - (0.7)(0.8)(0.6)(0.5) = 0.518.
\end{aligned}$$

Question 2. We do this in parts:

Part 1. I claim that it is possible to compute a unique value for $\mathbb{P}[A]$. I do so by computing this value. First, write

$$\begin{aligned}
\mathbb{P}[A \cup B] &= \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B] \\
\mathbb{P}[A \cup B^c] &= \mathbb{P}[A] + (1 - \mathbb{P}[B]) - \mathbb{P}[A \cap B^c].
\end{aligned}$$

Adding these two expressions,

$$\begin{aligned}
\mathbb{P}[A \cup B] + \mathbb{P}[A \cup B^c] &= 2\mathbb{P}[A] + 1 - (\mathbb{P}[A \cap B] + \mathbb{P}[A \cap B^c]) \\
\mathbb{P}[A \cup B] + \mathbb{P}[A \cup B^c] &= 2\mathbb{P}[A] + 1 - \mathbb{P}[A]
\end{aligned}$$

Thus,

$$\mathbb{P}[A] = \mathbb{P}[A \cup B] + \mathbb{P}[A \cup B^c] - 1 = 0.2.$$

Part 2. I claim that there is no unique value for A. In particular, let $\Omega = \{1,2,3,4\}$, $A = \{1,2\}$ and $B = \{1,3\}$. We define two probabilities P_1, P_2 on Ω that satisfy these equalities, but which satisfy $\mathbb{P}_1[B] \neq \mathbb{P}_2[B]$. Both these probabilities satisfy

$$\mathbb{P}_i[\{3\}] = \mathbb{P}[\{4\}] = 0.4.$$

However, we set

$$\begin{aligned}
\mathbb{P}_1[\{1\}] = 0,\ \mathbb{P}_1[\{2\}] = 0.2 \\
\mathbb{P}_2[\{1\}] = 0.1,\ \mathbb{P}_2[\{2\}] = 0.1
\end{aligned}$$

We then calculate that for $i \in \{1,2\}$,

$$\mathbb{P}_i[A \cup B] = 0.6$$

$$\mathbb{P}_i[A \cup B^c] = 0.6,$$

but

$$\mathbb{P}_1[B] = 0.4 \neq 0.5 = \mathbb{P}_2[B].$$

Multiple Choice

Question 1. We calculate

$$\begin{aligned} \mathbb{P}[\max(X_1, X_2, X_3, X_4, X_5) \geq 5] &= 1 - \mathbb{P}[\max(X_1, X_2, X_3, X_4, X_5) < 5] \\ &= 1 - \mathbb{P}[X_1, \ldots, X_5 \leq 4] \\ &= 1 - \mathbb{P}[X_1 \leq 4]^5 = 1 - (\frac{2}{3})^5 \approx 0.868. \end{aligned}$$

Question 2. We have

$$1 = \int f(x)dx = c \int_1^3 (1 + x^2)dx = \frac{32}{3}.$$

Thus,

$$c = \frac{3}{32} \approx 0.094.$$

Question 3. For $1 \leq i \leq 50$, let $X_i = 1$ if the i'th lottery ticket wins and 0 otherwise. Let $X = \sum_{i=1}^{50} X_i$. We know that X has binomial distribution with success probability p, and that

$$0.5 = 1 - \mathbb{P}[X = 0] = 1 - (1 - p)^{50}.$$

Thus,

$$1 - p = (0.5)^{\frac{1}{50}},$$

so $p \approx 0.014$.

Question 4. Let X be the number of customers arriving in the first 15 minutes. It is Poisson, with rate $\frac{5}{4} = 1.25$. Thus,

$$\mathbb{P}[X = 0] = e^{-1.25} \approx 0.287.$$

Question 5. For the random student, let F and S denote the events that the student had the flu and the flu shot respectively. Then

$$\begin{aligned} \mathbb{P}[S|F] &= \frac{\mathbb{P}[S \cap F]}{\mathbb{P}[F]} \\ &= \frac{\mathbb{P}[F|S]\mathbb{P}[S]}{\mathbb{P}[F|S]\mathbb{P}[S] + \mathbb{P}[F|S^c]\mathbb{P}[S^c]} \\ &= \frac{(0.1)(0.9)}{(0.1)(0.9) + (0.2)(0.1)} \\ &\approx 0.818. \end{aligned}$$

Question 6. By our formula,

$$\mathbb{P}[X = 5] = \mathbb{P}[X \leq 5] - \mathbb{P}[X < 5] = 0.$$

Question 7. For $i \in \{1, 2, \ldots, 8\}$, let Y_i be the lifetime of the i'th lightbulb and let $X_i = \mathbf{1}_{Y_i < 8}$ Then

$$\mathbb{P}[X_i = 1] = \mathbb{P}[Y_i < 8] = 1 - e^{-\frac{8}{3}} \approx 0.931.$$

Thus,

$$\mathbb{P}[\sum_{i=1}^{8} X_i = 6] = \binom{8}{6}(0.931)^6(1 - 0.931)^2 \approx 0.087.$$

Question 8. The probability of getting 4 cards of any particular value is:

$$\frac{\binom{48}{3}}{\binom{52}{7}}.$$

Since there are 13 values possible, the answer is

$$13\frac{\binom{48}{3}}{\binom{52}{7}} \approx 0.00168.$$

Added later: In the exam, there was some confusion as to whether I meant to include hands such as 2222333. I did (and I think this is the only reasonable interpretation of the word 'contain' in the question statement). However, so as to not disadvantage people who found this confusing, I also accept the answer to the very similar (and indeed slightly harder) question: *What is the probability that this 7-card hand contains a 'four-of-a-kind,' and that the remaining three cards are all distinct?*

I will not accept the answer to other related interpretations, as the question was clarified on the board during the exam and there were no further questions after this point.

The answer to the second question is:

$$\frac{(13)(48)(44)(40)}{\binom{52}{7}\, 3!} \approx 0.00138.$$

Note that this answer is still much closer to the correct answer than it is to any of the other options.

Statistics

The number of incorrect answers per multiple choice question were:

$$1 : 50$$
$$2 : 19$$
$$3 : 65$$
$$4 : 24$$
$$5 : 16$$
$$6 : 61$$
$$7 : 82$$

References

Each question on the midterm was quite similar to at least one question that we've seen already. I include some references for two reasons:

(1) To help with further studying, and
(2) So that you have an idea about what I mean when I say that questions are *quite similar*

Long Answer.

(1) This question is almost identical to Example 28 of the lecture notes (see also Example 124 from the exam review).
(2) The first part of this question is exactly Example 122 of the exam review. The second part is a cross between Example 122 and Example 123 of the exam review. **Note:** I considered this question to be tricky.

Multiple Choice.

(1) This is Example 103 of the lecture notes with a few numbers changed.
(2) This is Example 99 of the lecture notes with a few numbers changed.
(3) This is a 'standard' question asking for the parameter of a binomial distribution given a probability. This particular calculation also shows up in Example 96 of the lecture notes.
(4) This is Example 74 of the lecture notes with a few numbers changed.
(5) This is Example 36 of the lecture notes with a few numbers changed.
(6) This is Example 53 of the lecture notes with a few numbers changed.
(7) This is Example 97 of the lecture notes with a few numbers changed (and a $\leq$ sign changed into a $=$ sign; this does not significantly change the method for the solution). **Note:** This question is a compound question, and I expected it to be difficult.
(8) This question is similar to, but more difficult than, Question **2.1-15A** of Homework 2. **Note:** This was the only question on the exam that was more difficult than its 'model' question from the notes/homework. As such, I expected it to be difficult.

Comparison to previous class: I try to measure how well the class as a whole is doing compared to previous, comparable classes. This is useful to me (I care about grades being fair and somewhat accurate, and want to make sure that the new exam questions are comparable in difficulty to old exam questions). I share them because I think that some students may also find even this very rough information to be useful.

Here are the updates for the class so far. Two questions on the midterm exam were included to give a rough idea as to how the class as a whole is doing: The first long answer question and the 6'th multiple choice question. That is:

- Both questions were included (with a few numbers changed) on the midterm for the introductory probability class that I taught last year.
- Both questions (with a few numbers changed) were discussed in the midterm review.

- Both questions (with a few numbers changed) were also given as examples in class; both times, I mentioned that the questions were likely to appear on the final or midterm.

Compared to last year, the grades on the first of these questions was about 10 percent lower; the grades on the second question were about 45 percent lower. The average on the exam as a whole was about 10 percent lower before adjustment.