Name_____

Student Number_____

University of Calgary
Schulich School of Engineering
Fall 2008 Final Examination

ENGG407
Numerical Methods

December 12, 2008, 12-3pm

3 Hours Duration

1. Examination is closed book.
2. No calculators
3. You do not need to simplify the numerical expressions unless stated
4. All angles are in radians, for example sin(x), x is assumed to be in radians
5. Final exam counts for 45% of overall course grade.
6. Exam consists of 6 questions. Write answers in the space provided below each question.
7. Total marks for the exam is 100. Marks value of each question is indicated.

question	Area	value	Mark
1	**miscellaneous**	**15**	
2	**Root Finding and optimization**	**15**	
3	**Linear Equations**	**15**	
4	**Regression and Interpolation**	**20**	
5	**Integration**	**15**	
6	**Differential Equations**	**20**	
total		**100**	

EXAMINATION RULES AND REGULATIONS

STUDENT IDENTIFICATION

Each candidate must sign the Seating List confirming presence at the examination. All candidates for final examinations are required to place their University of Calgary I.D. cards on their desks for the duration of the examination. (Students writing mid-term tests can also be asked to provide identity proof.) Students without an I.D. card who can produce an acceptable alternative I.D., e.g., one with a printed name and photograph, are allowed to write the examination.

A student without acceptable I.D. will be required to complete an Identification Form. The form indicates that there is no guarantee that the examination paper will be graded if any discrepancies in identification are discovered after verification with the student's file. A Student who refuses to produce identification or who refuses to complete and sign the Identification Form is not permitted to write the examination.

EXAMINATION RULES

(1) Students late in arriving will not normally be admitted after one-half hour of the examination time has passed.

(2) No candidate will be permitted to leave the examination room until one-half hour has elapsed after the opening of the examination, nor during the last 15 minutes of the examination. All candidates remaining during the last 15 minutes of the examination period must remain at their desks until their papers have been collected by an invigilator.

(3) All inquiries and requests must be addressed to supervisors only.

(4) Candidates are strictly cautioned against:

(a) speaking to other candidates or communicating with them under any circumstances whatsoever;

(b) bringing into the examination room any textbook, notebook or memoranda not authorized by the examiner;

(c) making use of calculators and/or portable computing machines not authorized by the instructor;

(d) leaving answer papers exposed to view;

(e) attempting to read other student's examination papers.

The penalty for violation of these rules is suspension or expulsion or such other penalty as may be determined.

(5) Candidates are requested to write on both sides of the page, unless the examiner has asked that the left hand page be reserved for rough drafts or calculations.

(6) Discarded matter is to be struck out and not removed by mutilation of the examination answer book.

(7) Candidates are cautioned against writing in their answer book any matter extraneous to the actual answering of the question set.

(8) The candidate is to write his/her name on each answer book as directed and is to number each book.

(9) A candidate must report to a supervisor before leaving the examination room.

(10) Answer books must be handed to the supervisor-in-charge promptly when the signal is given. Failure to comply with this regulation will be cause for rejection of an answer paper.

(11) If during the course of an examination a student becomes ill or receives word of a domestic affliction, the student should report at once to the supervisor, hand in the unfinished paper and request that it be cancelled. If physical and/or emotional ill health is the cause, the student must report at once to a physician/counsellor

so that subsequent application for a deferred examination is supported by a completed Physician/Counsellor Statement form. Students can consult professionals at University Health Services or University Counselling Services during normal working hours or consult their physician/counsellor in the community.

Should a student write an examination, hand in the paper for marking, and later report extenuating circumstances to support a request for cancellation of the paper and for another examination, such a request will be denied.

(12) Smoking during examinations is strictly prohibited.

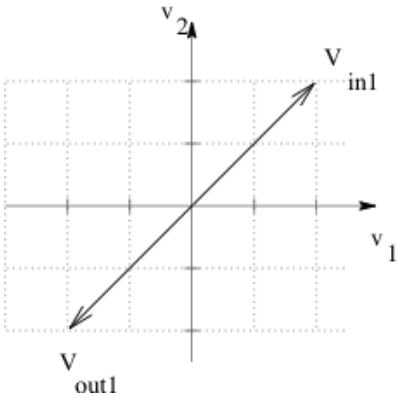
RO 94-01

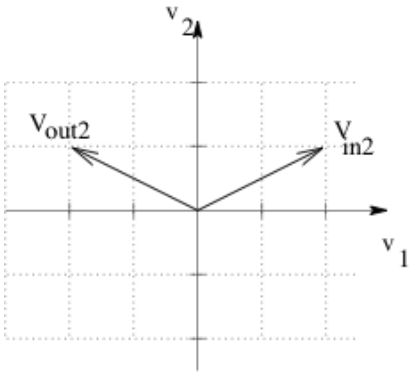
QUESTION 1:

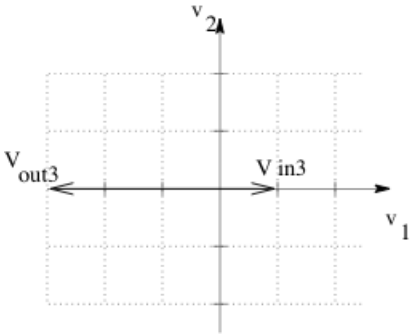
A) A continuous differentiable function, *f(x),* is known to have a single optimum corresponding to a function maximum point between x=3 and x=8. Can the false position root finding method be used to find this maximum? Why or why not? If so how? **[2 marks]**

B) The numerical integration of $f(x)$ from $x = A$ to $x = B$ with $B > A$ is considered with the 1/3 and 3/8 Simpsons rules. What characteristic of $f(x)$ will ensure that the results will be exactly the same? **[3 marks]**

C) A system with two inputs and two outputs is described by a 2x2 matrix. Three different input vectors, $V_{in_1}, V_{in_2}, V_{in_3}$ and their corresponding output vectors $V_{out_1}, V_{out_2}, V_{out_3}$ are shown below. Explain which one(s) of the inputs can be an eigenvector and state what the corresponding eigenvalue is. [3 marks]







D) The 2nd order Taylor series expansion of function $f(x) = \left| xe^{x} \right|$ is to be created with an expansion point of x_0. Consider each of the possible expansion points given below and determine which can be used to create a valid Taylor expansion. **[3 marks]**

i. $x_0 = 1$

ii. $x_0 = -1$

iii. $x_0 = 0$

E) A differential equation is of the form $\frac{dy}{dx} = C$ where C is a constant. Does Heun's method provide any advantage over using the forward Euler in this case? Why or why not? **[2 marks]**

F) What is meant by a stiff ODE? **[2 marks]**

Question 2: Root Finding

A) Consider the function $f(x) = x + \exp(x)$ where x is complex and represented as $x = a + ib$. It is desired to find the complex root that satisfies this equation using the Newton Raphson method. This is done by representing $f(x)$ as a first order Taylor series around $x = 0$ in terms of real and imaginary components a and b. Determine the expressions for the Taylor series expansion of $f(x)$ in terms of the real and imaginary functions, $f(x) = f(a+ib) = U(a,b) + jV(a,b)$, around zero. **[4 marks]**

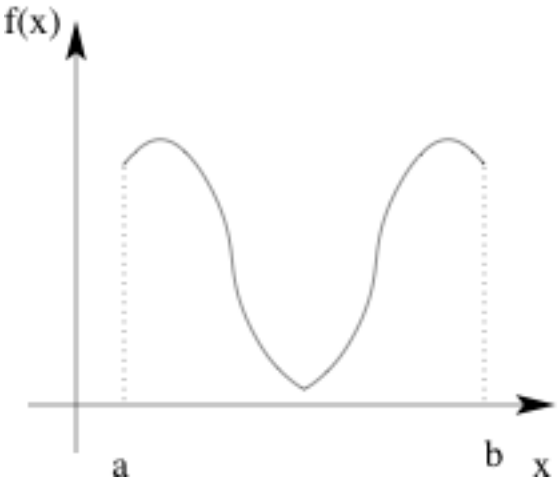
B) For the first order Taylor series expansion of function *f(x)* given in **part A,** write the Newton Raphson update obtained from the starting point (a_0, b_0) for finding the complex root(s) of *f(x)*. **[4 marks]**

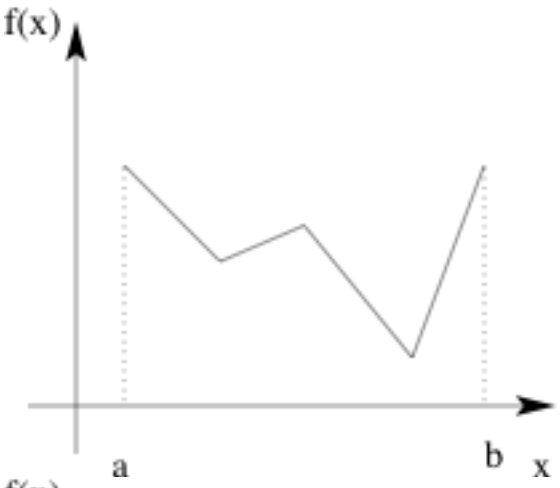
C) A function $f(\vec{x}) = \vec{x}^T C\vec{x} + b^T\vec{x} + d$ is given where

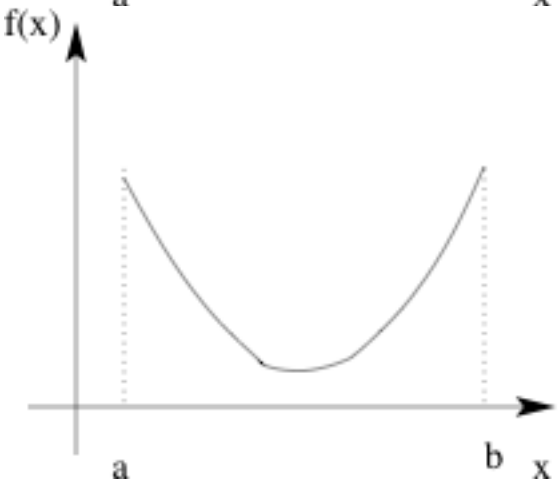
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}, b = \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix}, d = 300$$

Find the minimum(s) of $f\left(\vec{x}\right)$ by using the derivative method. **[4 marks]**

D) It is desired to find the minimums of the three functions that are shown in the figure below. For each function determine which method, bisection with [a, b] as the bracket or Newton Raphson's method with either *a* or *b* as a starting point, is best suited for finding the minimum point. Another possibility is that neither of the methods are applicable. Explain your answer for each case.
[3 marks]







Question 3: Linear Equations

Consider the following matrix A

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 4 & 1 & 1 \\ 0 & 1 & 9 \end{bmatrix}$$

A) Determine the LU decomposition of the matrix A. Pivoting is not required. **[5 marks]**

B) Consider a matrix

$$A = \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$$

Determine the eigenvalues and the corresponding eigenvectors of A. **[10 marks]**

Question 4: Regression and Interpolations

Data is produced from a certain measurement which is given in the table below:

Time t	Measurements	
	x	y
0	2	0
1	5	1
2	10	1
4	25	2
5	33	1

Use this data to answer the following questions:

A) PhD candidate Fred states that *x* and *y* are independent and the relationship of $x(t)$ is given as:

$$x(t) = C\sin(t) + Bt + E$$

Set up the matrix equation to solve for the unknown coefficients C, B and E based on least squares regression assuming the date fits Fred's postulate, and show how the problem is solved. (You do not have to solve the resulting matrix equation.) **[4 marks]**

B) His supervisor Dr. George disagrees and says that *x* and *y* are dependent and the data is related as

$$x(t) + \exp(y(t)) = C\sin(t) + Bt + E$$

Show how Dr. George could solve for the unknown coefficients C, B and E based on the least squares regression of his postulated model. **[4 marks]**

C) Dr George and Fred decide to resolve there disagreement by finding which postulate gives the best fit to the data by computing the total square error. Show how this is done. **[2 marks]**

E) Three data points are given in the table below. It is desired to interpolate these points using the two Splines, f_1, f_2 .

a. Find the Splines if f_1 is the quadratic Spline and f_2 is the linear Spline. **[3 marks]**

b. Again find the Splines if f_1 is the linear Spline and f_2 is the quadratic Spline. **[3 marks]**

c. Which one of these interpolations can be used for finding the optimum point of the Interpolated function using Newton Raphson technique and starting from x = 0 and find this optimum point. **[4 marks]**

x	*f(x)*
0	3
1	1
2	2

Question 5: Integration

A) Numerically integrate the function $f(x) = \exp(2x)$ over the range of $1 \le x \le 2$ using the two point-Gauss Quadrature method. Assume that two equal width panels are used over this interval. **[3 marks]**

B) Numerically integrate the function $f(x) = \exp(2x)$ over the range of $1 \le x \le 2$ using the three-point Gauss Quadrature method with a single integration panel. **[2 marks]**

C) A function $f(x, y, z)$ has a second order Taylor expansion at an expansion point of $\{x_0 = 0 \quad y_0 = 0 \quad z_0 = 0\}$. Assume that $f(0,0,0) = 3$. The Jacobian and Hessian of $f(x, y, z)$ are given as

$$J = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Determine the approximate integration of $\int_0^1 \int_0^1 \int_0^1 f(x, y, z)\, dxdydz$ based on the second order Taylor expansion around (0,0, 0). **[5 marks]**

C) A function is evaluated at 6 equally spaced points as given in table below. Separate the function into two non-overlapping panels and use the Simpson $\frac{3}{8}$ rule to find an approximation for the integral for the first panel and the Simpson $\frac{1}{3}$ to calculate an approximation to the integral of the function. **[5 marks]**

i	*x*	*f(x)*
0	0	3
1	1	1.5
2	2	1
3	3	2
4	4	2
5	5	2

Question 6: Differential Equations

A) An electric circuit consists of a capacitor, inductor and resistor connected in series. The voltage across the capacitor is a function of time denoted by $v(t)$ which is related to the generator voltage, $g(t)$, by the differential equation

$$A\frac{d^2v(t)}{dt^2}+\left(B+Ct\right)\frac{dv(t)}{dt}+Dv(t)=g(t)$$

Assume that A, B C and D are known constants and that $g(t)$ is the given independent forcing function. Set up the 2nd order DEQ as a first order DEQ in matrix form.
[5 marks]

B) In part A, assume that at t=0, $\left\{ v(t) = 1 \quad \frac{dv(t)}{dt} = 2 \right\}$. Determine the steps required to determine the values of $\left\{ v(t) \quad \frac{dv(t)}{dt} \quad \frac{d^2v(t)}{dt^2} \right\}\Big|_{t=\Delta t}$ using the forward Euler method with a single step time increment of Δt. **[5 marks]**

C) Now give the update for Heuns method. **[5 marks]**

D) Consider the DEQ given as

$$\frac{dy}{dx} = y^2 \exp(x)$$

which has an initial value of $\{x = 0, y = 1\}$. It is desired to find the function of $y(x)$ over the range of $1 \le x \le 2$. Instead of solving the DEQ by using standard numerical ways of solving the DEQ, show how this can be done with integration instead. **[5 marks]**

Aid Sheet

Taylor expansion Formulas:

$$\widehat{f_N}(x) = \sum_{n=0}^{N} \frac{1}{n!} f^{(n)}(x_o)(x - x_o)^n$$

$$\hat{f}_1(\mathbf{x}) = f(\mathbf{x}_0) + J^T(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$$

$$\hat{f}_2(\mathbf{x}) = f(\mathbf{x}_0) + J^T(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + \frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^T H(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$$

Jacobian and Hessian:

$$J^T(x_{1,0}, x_{2,0}, ..., x_{N,0}) = \begin{bmatrix} f^{(x_1)}(x_{1,0}, x_{2,0}, ..., x_{N,0}) \\ f^{(x_2)}(x_{1,0}, x_{2,0}, ..., x_{N,0}) \\ \vdots \\ f^{(x_N)}(x_{1,0}, x_{2,0}, ..., x_{N,0}) \end{bmatrix} \quad \mathbf{H}(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_N} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_N \partial x_N} \end{bmatrix}$$

Integration

Simpsons rule per panel: $I = \frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2))$, $I = \frac{3h}{8}(f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3))$

Gauss Quadrature integration per panel from x=-1 to x=1 is given by

$I = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$, $I = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$

DEQ's

Fourth order Runge Kutta $y_{i+1} = y_i + \frac{h}{6}(k_1 + k_2 + k_3 + k_4)$

$$k_1 = f(t_i, y_i)$$
$$k_2 = f\left(t_i + \frac{h}{2}, y_i + \frac{k_1 h}{2}\right)$$
$$k_3 = f\left(t_i + \frac{h}{2}, y_i + \frac{k_2 h}{2}\right)$$
$$k_4 = f(t_i + h, y_i + k_3 h)$$

Heuns $y_{i+1} = y_i + \frac{\Delta x}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_i + \Delta x f(x_i, y_i))\right)$

Miscellaneous

$\exp(jx) = \cos(x) + j\sin(x)$

Lagrange polynomial $L_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{(x - x_j)}{(x_i - x_j)}$