Chapter 23: Diagonalizability

Recall that:

Let A be an $n \times n$ matrix. If $\lambda \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$ is a non-zero vector such that

$$A\vec{x} = \lambda \vec{x},$$

then $\vec{x}$ is called an eigenvector of A and λ its corresponding eigenvalue.

* The eigenvalues of A are the roots of the characteristic polynomial $\det(A - \lambda I)$ and the multiplicity of λ as a root is called its "algebraic multiplicity".

Example: The characteristic polynomial of $A = \begin{pmatrix} 16 & 2 & 17 \\ 0 & -43 & 14 \\ 0 & 0 & 16 \end{pmatrix}$ is $\det(A - \lambda I) = -(\lambda - 16)^2(\lambda - 43)$. Hence, the eigenvalues of A are $\lambda = -43$ (with algebraic multiplicity of 1) and $\lambda = 16$ (with algebraic multiplicity of 2).

* The λ-eigenspace of A is given by

$$E_\lambda = \mathrm{Null}(A - \lambda I) = \{\vec{x} \in \mathbb{R}^n \mid (A - \lambda I)\vec{x} = \vec{0}\} = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda \vec{x}\},$$

whose non-zero elements are the eigenvectors of A corresponding to the eigenvalue λ.

Definition: The dimension of the eigenspace E_λ is called the "geometric multiplicity" of λ.

Theorem (23.1): (Limits of geometric multiplicity)

Let λ be an eigenvalue of A. Then the geometric multiplicity of λ is at least 1, and at most equal to the algebraic multiplicity of λ. That is

$$\boxed{1 \leqslant \dim(E_\lambda) \leqslant \text{algebraic multiplicity of } \lambda}$$

Definition: The $n \times n$ matrix A is said to be "diagonalizable", if there is a basis of $\mathbb{R}^n$ consisting entirely of eigenvectors of A.

Example: Let $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$. Find all eigenvalues and a basis for each eigenspace and check if A is diagonalizable.

$$\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{pmatrix}, \text{ and also}$$

$$\begin{pmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{pmatrix} \underset{-2R_2+R_3 \to R_3}{\sim} \begin{pmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 0 & 2+2\lambda & -1-\lambda \end{pmatrix}$$

$$\underset{2C_3+C_2 \to C_2}{\sim} \begin{pmatrix} 3-\lambda & 10 & 4 \\ 2 & 4-\lambda & 2 \\ 0 & 0 & -1-\lambda \end{pmatrix}$$

by recalling the fact that the determinant won't change under elementary row and column operation, we can do the cofactor expansion on the third row of the last obtained matrix and so:

$$\det(A-\lambda I) = \det\begin{pmatrix} 3-\lambda & 10 & 4 \\ 2 & 4-\lambda & 2 \\ 0 & 0 & -1-\lambda \end{pmatrix} = -(\lambda+1)\left((3-\lambda)(4-\lambda)-20\right)$$

$$= -(\lambda+1)^2(\lambda-8) \quad : \text{characteristic polynomial of } A.$$

eigenvalues: $\lambda = -1$ of algebraic multiplicity 2,

$\lambda = 8$ of algebraic multiplicity 1.

let $\lambda = 8$:

$$E_8 = \mathrm{Null}(A-8I) = \begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \sim \cdots \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \dim(E_8) = \dim(\mathrm{Null}(A-8I)) = 1$: geometric multiplicity of 8

$\left(= \# \text{ parameters in } (A-8I) \right)$.

$\Rightarrow$ A basis for E_8 would be :

the general solution to $\mathrm{Null}(A-8I) = E_8$ would be as

$$\left\{ \begin{pmatrix} t \\ t/2 \\ t \end{pmatrix} \middle| \, t \in \mathbb{R} \right\} \Rightarrow E_8 = \{(1, 1/2, 1)\}.$$

Next, let $\lambda = -1$:

$$E_{-1} = \mathrm{Null}(A+I) = \begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\to \dim(E_{-1}) = \dim(\cancel{A} \text{ Null}(A+I)) = \#$ parameters in the general solution to $(A+I)\vec{x} = \vec{0}$ $= 2$: geometric multiplicity of -1.

As is seen, the geometric and algebraic multiplicities of $\lambda = -1$ are equal. A basis for (-1)-eigenspace would be

$$E_{-1} = \{(-\tfrac{1}{2}, 1, 0), (-1, 0, 1)\}.$$

Finally, the set $\{(1, \tfrac{1}{2}, 1), (-\tfrac{1}{2}, 1, 0), (-1, 0, 1)\}$ is LI and hence a basis of $\mathbb{R}^3$ and since $\mathbb{R}^3$ has a basis consisting entirely of eigenvectors of A, we conclude that $\underline{A}$ is diagonalizable.

Let's construct a matrix P whose columns are an eigenvector basis corresponding to A:

$$P = \begin{pmatrix} 1 & -\frac{1}{2} & -1 \\ \frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \text{ which is invertible (why?)}$$

Let's call those columns $\vec{v}_i$ for short. Then,

$$AP = A(\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3) = (8\vec{v}_1 \quad (-1)\vec{v}_2 \quad (-1)\vec{v}_3)$$

$$= (\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3)\begin{pmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = PD,$$

where D is the diagonal matrix containing eigenvalues of A.

Since P is invertible,

$$\boxed{P^{-1}AP = D, \quad \text{or} \quad A = PDP^{-1}}$$

Proposition: If P is a matrix whose columns are an eigenvector basis of $\mathbb{R}^n$ corresponding to A and D is the diagonal matrix whose diagonal entries are the corresponding eigenvalues, then

$$\boxed{P^{-1}AP = D, \quad \text{or} \quad A = PDP^{-1}}$$

Example: use A from the previous example and find A^n, for any n:

$A = PDP^{-1} \Rightarrow$ for instance $n=5$

$$A^5 = AAAAA = (PDP^{-1})(PDP^{-1})(PDP^{-1})(PDP^{-1})(PDP^{-1})$$

$$= PD^5P^{-1}$$

$$D^5 = \begin{pmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}^5 = \begin{pmatrix} 8^5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

So, in general $A^n = PD^nP^{-1}$, where $D^n = \begin{pmatrix} 8^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & (-1)^n \end{pmatrix}$

Note: The equation $A = PDP^{-1}$ is why we call this process "diagonalization".

Failure of Diagonalization:

We have seen that real matrices could have complex eigenvalues and so could be diagonalizable over $\mathbb{C}$, without being diagonalizable over $\mathbb{R}$.

A more seriou~~ps~~ problem occurs when there is a deficiency in the geometric multiplicity of one or more eigenvalues:

Example: Is $A = \begin{pmatrix} 2 & -4 & -1 \\ 0 & -18 & -4 \\ 0 & 80 & 18 \end{pmatrix}$ diagonalizable?

$$\det \begin{pmatrix} 2-\lambda & -4 & -1 \\ 0 & -18-\lambda & -4 \\ 0 & 80 & 18-\lambda \end{pmatrix} = \cdots = -(\lambda-2)^2(\lambda+2)$$

$$\Rightarrow \text{eigenvalues: } \begin{cases} \lambda = 2 \ (\text{algebraic multiplicity} = 2) \\ \lambda = -2 \ (\text{algebraic multiplicity} = 1) \end{cases}$$

For the eigenspace E_2, let's row reduce $A - 2I$:

$$\begin{pmatrix} 0 & -4 & -1 \\ 0 & -20 & -4 \\ 0 & 80 & 16 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1/4 \\ 0 & 1 & 1/5 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow x_1$: non-leading variable $\Rightarrow$ basic solution $(1, 0, 0)$

and so $\dim(E_2) = 1$. Since geometric multiplicity of $\lambda = 2$ is not equal to its algebraic multiplicity, then we conclude that A is not diagonalizable.