Chapter 22: Eigenvalues and Eigenvectors

Definition: Let A be an $n \times n$ matrix. If $\lambda \in \mathbb{R}$ is a scalar and $\vec{x} \in \mathbb{R}^n$ is a "non-zero" vector such that

$$\boxed{A\vec{x} = \lambda \vec{x}},$$

then $\vec{x}$ is called an "eigenvector" of A and λ is its corresponding "eigenvalue".

Example:

(1) Let $A = \begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix}$, then

$A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \lambda = 1$ (eigenvalue) & $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is the corresponding eigenvector.

$A\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 4\begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow \lambda = 4$ (eigenvalue) & $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is the corresponding eigenvectors.

(2) Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \Rightarrow A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \vec{0}$

Note: The matrix A can have 0 as an eigenvalue but the vector $\vec{0}$ is never an eigenvector.

Application : suppose $\{\vec{x}_1, \ldots, \vec{x}_n\}$ is a basis of $\mathbb{R}^n$ and are the eigenvectors of the matrix A, with the corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. Then for any $\vec{v} = a_1\vec{x}_1 + \cdots + a_n\vec{x}_n$, we have

$$\begin{aligned} A\vec{v} &= A(a_1\vec{x}_1 + \cdots + a_n\vec{x}_n) \\ &= a_1(A\vec{x}_1) + \cdots + a_n(A\vec{x}_n) \\ &= a_1(\lambda_1\vec{x}_1) + \cdots + a_n(\lambda_n\vec{x}_n), \end{aligned}$$

or, we just have to scale each coordinate (coefficient) by appropriate λ_i.

⊛ Finding eigenvalues of A:

$$A\vec{x} = \lambda\vec{x} \iff A\vec{x} = \lambda I\vec{x} \iff A\vec{x} - \lambda I\vec{x} = \vec{0} \iff (A - \lambda I)\vec{x} = \vec{0}$$

Two Cases:

(1) If $(A - \lambda I)\vec{x} = \vec{0}$ has a unique solution, then there are no eigenvectors corresponding to λ (or, λ is not an eigenvalue).

(2) If $(A - \lambda I)\vec{x} = \vec{0}$ has infinitely many solutions, then any non-trivial solution ($\vec{x} \neq \vec{0}$) is an eigenvector and so λ is an eigenvalue.

The second case implies that:

λ is an eigenvalue of $A \iff$ the matrix $A - \lambda I$ is not invertible

λ is an eigenvalue of $A \iff \det(A - \lambda I) = 0$.

Definition: The polynomial $\det(A-\lambda I)$ is called the "characteristic polynomial" of A.

Example: Find all the eigenvalues of $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$:

$$A - \lambda I = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{pmatrix} \Rightarrow$$

$$\det(A-\lambda I) = (1-\lambda)(1-\lambda) - 9 = \lambda^2 - 2\lambda - 8 = (\lambda-4)(\lambda+2) = 0$$

$\Leftrightarrow \lambda = 4$ or $\lambda = -2$ $\Rightarrow$ eigenvalues are $\underline{4}$ and $\underline{-2}$.

(check that $A-4I$ and $A+2I$ are not invertible).

⊛ **Finding the eigenvectors of A:**

Recall that the eigenvectors are the non-trivial solutions $\vec{x}$ of $(A-\lambda I)\vec{x} = \vec{0}$.

In other words, the eigenvectors of associated to λ are the non-zero vectors in $\text{Ker}(A-\lambda I) = \text{Null}(A-\lambda I)$.

Definition: Let λ be an eigenvalue of A. Then, the subspace

$$E_\lambda = \text{Null}(A-\lambda I) = \{\vec{x} \in \mathbb{R}^n \mid (A-\lambda I)\vec{x} = \vec{0}\} = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x}\}$$

is called the "$\underline{\lambda\text{-eigenspace}}$" of A. Its non-zero elements are the eigenvectors of A corresponding to the eigenvalue λ.

So, instead of eigenvectors of A, we can look for bases of eigenspaces of the matrix A.

Example: Find a basis for each eigenspace of $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$.

Recall that the eigenvalues are 4 & -2.

(1) $\lambda = 4 \Rightarrow A - \lambda I = A - 4I = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$,

So, the general solution to the $(A-4I)\vec{x} = \vec{0}$ is

$E_4 = \text{Null}(A-4I) = \{(r,r) \mid r \in \mathbb{R}\} = \{(1,1)\}$

(2) $\lambda = -2 \Rightarrow A - \lambda I = A + 2I = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$,

So, the general solution to $(A+2I)\vec{x} = \vec{0}$ is

$E_{-2} = \text{Null}(A+2I) = \{(-r,+r) \mid r \in \mathbb{R}\} = \{(-1,+1)\}$

Theorem (22.1): (Independence of Eigenvectors)

Let A be an $n \times n$ matrix. Then, any set consisting of eigenvectors of A corresponding to distinct eigenvalues is linearly independent (LI).

Summary:

(1) The characteristic polynomial of an $n \times n$ matrix A is a degree n polynomial, so it has at most n distinct roots.

(2) Each root of the characteristic polynomial is an eigenvalue of A.

(3) Each eigenvalue of A gives an eigenspace of dimension at least 1.

(4) So, if the characteristic polynomial has exactly n distinct roots, then we can choose one eigenvector for each eigenspace and thus produce a basis of $\mathbb{R}^n$ consisting entirely of eigenvectors of A.

Definition: The $n \times n$ matrix A is said to be "diagonalizable" if there is a basis of $\mathbb{R}^n$ consisting entirely of eigenvectors of A.

Remark: If an $n \times n$ matrix A has n distinct real eigenvalues, then A is diagonalizable.

Problematic cases:

(1) Not enough real roots:

Example: $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \Rightarrow \det(A - \lambda I) =$

$$= \det\begin{pmatrix} \cos\theta - \lambda & \sin\theta \\ -\sin\theta & \cos\theta - \lambda \end{pmatrix} = (\lambda - \cos\theta)^2 + \sin^2\theta$$

$$= \lambda^2 - 2\cos(\theta)\lambda + \cos^2\theta + \sin^2\theta = \lambda^2 - 2\lambda\cos\theta + 1.$$

(using quadratic formula)

$$\lambda = \frac{1}{2}\left(2\cos\theta \pm \sqrt{4\cos^2\theta - 4}\right) = \cos\theta \pm \sqrt{-\sin^2\theta} = \cos\theta \pm i\sin\theta.$$

which are complex numbers unless $\theta = 0$ or π and so the eigenvectors would have complex coordinates (they are in $\mathbb{C}^2$).

Note: Eigenvalues can be complex numbers. In this case, the eigenvectors are not a basis of $\mathbb{R}^n$ and A is not diagonalizable over $\mathbb{R}$.

(2) Not enough eigenvectors:

If λ is an eigenvalue, then $\dim(E_\lambda) \geq 1$. But if λ is a repeated root of the characteristic polynomial, with multiplicity k, then we'd need $\dim(E_\lambda) = k$. which doesn't always happen:

<u>Example</u>: Find the eigenvalues of $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ and a basis for each eigenspace:

characteristic polynomial: $(\lambda-3)^2 = 0 \Rightarrow \lambda = 3$ (multiplicity 2)

but $\operatorname{rank}(A-3I) = 1 \Rightarrow \dim(E_3) = \dim(\operatorname{Null}(A-3I)) = 2-1 = 1$

we only got a 1-dimensional subspace of eigenvectors, which makes it impossible to find an eigenvector basis of $\mathbb{R}^2$. A isn't diagonalizable

<u>Note</u>: If the characteristic polynomial has a repeated root, then it can happen that we don't have enough LI eigenvectors of A to from a basis.

<u>Exercise</u>: $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$. Find all eigenvalues and a basis for each eigenspace and decide if A is diagonalizable.