Chapter 21: Determinants

Let A be an $n \times n$ square matrix:

(1) $n=1 \Rightarrow \det[a] = a$,

(2) $n=2 \Rightarrow \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$,

(A is invertible $\Leftrightarrow \det(A) \neq 0$); geometrically it is the absolute value of the area of the parallelogram whose vectors are the rows of A.

(3) $n=3 \Rightarrow \det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix} = a\det\begin{pmatrix} e & f \\ h & j \end{pmatrix} - b\det\begin{pmatrix} d & f \\ g & j \end{pmatrix} + c\det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$

$= a(ej - fh) - b(dj - fg) + c(dh - ge)$

(cofactor expansion along the first row).

geometrically, it equals to the absolute value of the volume of the parallelipiped, given by the scalar triple product of the rows of A as vectors $\vec{u}, \vec{v}$ and $\vec{w}$, or $\vec{u}\cdot(\vec{v}\times\vec{w})$.

Definition: Let $A = [a_{ij}]$, then

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + \cdots + (-1)^{1+n} a_{1n}\det(A_{1n}),$$

where A_{ij} is the $(n-1)\times(n-1)$ matrix obtained from A by deleting row i and column j.

Example:
$$A = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\det(A) = 2\det\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - 3\det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + 4\det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
$$- 5\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= 2(1) - 3(-1) + 4(0) - 5(1) = 0$$

Theorem (21.1): "cofactor expansion along any row or column"

Let $A = [a_{ij}]$, then the determinant can be calculated via the "cofactor expansion along row i, for any i:

$$\det(A) = (-1)^{i+1} a_{i1} \det(A_{i1}) + (-1)^{i+2} a_{i2} \det(A_{i2}) + \cdots +$$
$$(-1)^{i+n} a_{in} \det(A_{in}),$$

and similarly, the determinant equals the "cofactor expansion along column j, for any j:

$$\det(A) = (-1)^{1+j} a_{1j} \det(A_{1j}) + (-1)^{2+j} a_{2j} \det(A_{2j}) + \cdots +$$
$$(-1)^{n+j} a_{nj} \det(A_{nj})$$

Example:
$$\det\begin{pmatrix} 2 & 3 & 4 \\ 1 & 0 & 3 \\ 2 & 0 & 4 \end{pmatrix} = 2\det\begin{pmatrix} 0 & 3 \\ 0 & 4 \end{pmatrix} - 3\det\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} + 4\det\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$$
$$= 6.$$

Fact: properties of the determinant

Let A be an $n \times n$ matrix:

(1) If A has a row or column of zeros, then $\det(A) = 0$,

(2) $\det(A) = \det(A^T)$.

Fact (21.2): Determinant of triangular matrices

The determinant of a triangular matrix is the product of the diagonal entries.

Example:

$$\det\begin{pmatrix} 2 & 2 & 4 & 7 & 6 \\ 0 & -3 & 7 & 1 & 3 \\ 0 & 0 & 1 & 12 & -8 \\ 0 & 0 & 0 & -2 & 21 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} = 2\det\begin{pmatrix} -3 & 7 & 1 & 3 \\ 0 & 1 & 12 & -8 \\ 0 & 0 & -2 & 21 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$= 2(-3)\det\begin{pmatrix} 1 & 12 & -8 \\ 0 & -2 & 21 \\ 0 & 0 & 3 \end{pmatrix} = 2(-3)(1)\det\begin{pmatrix} -2 & 21 \\ 0 & 3 \end{pmatrix}$$

$$= 2(-3)(1)(-2)(3)$$ = product of diagonal entries.

Note: If A is in RREF, then either

(1) A has rank n and the determinant is one,

(2) A has rank less than n and the determinant is zero.

Theorem 21.2: (Effect of row reduction on the determinant)

Let A be an $n \times n$ matrix and suppose you do the elementary row operation and obtain $\tilde{A}$. Then,

(1) If the row operation was interchange of two rows, then $\boxed{\det(A) = -\det(\tilde{A})}$

(2) If the row operation was multiplying a row by an scalar r, then

$$\boxed{\det(\tilde{A}) = r \det(A).}$$

(3) If the row operation was add a multiple of one row to another, then

$$\boxed{\det(\tilde{A}) = \det(A)}$$

***Remark**: The theorem above remains true if "row" is replaced by "Column".

Example: Find $\det(A)$, where $A = \begin{pmatrix} 2 & 1 & 3 & 5 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 3 & 1 & 4 \end{pmatrix}$.

$$A \underset{R_1 \leftrightarrow R_2}{\sim} \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 3 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 3 & 1 & 4 \end{pmatrix} \underset{-3R_3 + R_4 \to R_4}{\overset{-2R_1 + R_2 \to R_2}{\sim}} \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -3 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -5 & -5 \end{pmatrix}$$

$$\underset{\substack{-\frac{1}{3} R_3 \\ -\frac{1}{5} R_4}}{\overset{R_2 \leftrightarrow R_3}{\sim}} \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \underset{\substack{-R_2 + R_3 \to R_3 \\ R_3 + R_4 \to R_4}}{\sim} \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & -3 \end{pmatrix} = \tilde{A}$$

$$\Rightarrow \det(\tilde{A}) = (-1)\left(\frac{-1}{3}\right)\left(\frac{-1}{5}\right)(-1)\det(A) \Rightarrow \det(A) = 45.$$

⊛ properties of determinants:

Theorem 21.3: (Properties of the determinants)

Let A and B be $n \times n$ matrices. Then,

(1) $\det(rA) = r^n \det(A)$, for any $r \in \mathbb{R}$,

(2) $\det(AB) = \det(A)\det(B)$,

(3) $\det(A) = 0 \iff A$ is not invertible,

(4) If A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

proof: Please refer to the primary text.

Example: For which values of $\underline{\underline{c}}$ does $A = \begin{pmatrix} 1 & 0 & -c \\ -1 & 3 & 1 \\ 0 & 2c & -4 \end{pmatrix}$ have an inverse?

$$\det\begin{pmatrix} 1 & 0 & -c \\ -1 & 3 & 1 \\ 0 & 2c & -4 \end{pmatrix} \underset{\Downarrow}{=} \det\begin{pmatrix} 1 & 0 & 0 \\ -1 & 3 & 1-c \\ 0 & 2c & -4 \end{pmatrix} = 2(2+c)(c-3),$$

c times column 1 to column 3

$\Rightarrow \det A = 0$ if $c = -2$ or $c = 3$ and A has an inverse if $c \neq -2$ & $c \neq 3$.

Example: If $\det(A) = 2$, and $\det(B) = 5$, calculate $\det(A^3 B^{-1} A^T B^2)$.

$$\det(A^3 B^{-1} A^T B^2) = \det(A^3)\det(B^{-1})\det(A^T)\det(B^2)$$

$$= (\det A)^3 \frac{1}{\det(B)} \det(A) (\det B)^2$$

$$= 8 \cdot \frac{1}{5} \cdot 2 \cdot 25 = \boxed{80}$$