Chapter 18 : "Matrix Inverses"

(Throughout this chapter, A is a square $n \times n$ matrix).

Goal : Looking for a matrix A^{-1}, so that from $A\vec{x} = \vec{b}$ we can deduce

$$\boxed{\vec{x} = A^{-1}\vec{b}}$$

Definition : If A is an $n \times n$ matrix and B is an $n \times n$ matrix such that

$$AB = BA = I_n ,$$

then B is called an "inverse" of A and is denoted by $B = A^{-1}$. In this case, A is called invertible.

- Recall I_n is the $n \times n$ identity matrix.

Example :

(1) $I_n^{-1} = I_n$, Since $I_n^{-1} I_n = I_n I_n = (I_n)^2 = I_n$.

(2) $A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$. Then,

$$AB = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 ,$$

$$BA = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 ,$$

So, $B = A^{-1}$ and $A = B^{-1}$!

* Finding the inverse of a <u>2x2 matrix</u>:

Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $ad-bc \neq 0$, then

$$\boxed{A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}},$$

If $ad-bc=0$, then A is not invertible, for instance $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is not invertible, since if $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ were an inverse of A, then

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ a+c & b+d \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for every $a, b, c, d \in \mathbb{R}$. So, no inverse exists.

- Recall: If A is invertible, then $A\vec{x} = \vec{0}$ has a unique solution.

- <u>Lemma</u>: Suppose A is an invertible <u>$n \times n$</u> matrix. Then any linear system $A\vec{x} = \vec{b}$: (A is the <u>$n \times n$</u> coefficient matrix)

(1) is consistent, and

(2) has a unique solution.

<u>Proof</u>:

Consistency: $A(A^{-1}\vec{b}) = (AA^{-1})\vec{b} = I_n\vec{b} = \vec{b}$, so $\vec{x} = A^{-1}\vec{b}$ is a solution.

uniqueness: let's $\vec{y}$ be another solution, then $A\vec{y} = \vec{b} \Rightarrow A^{-1}A\vec{y} = A^{-1}\vec{b}$

$\Rightarrow \vec{y} = A^{-1}\vec{b}$, which is the solution, so $\vec{x} = \vec{y}$.

$\square$

$*$ Algebraic Properties of inverses:

If $k \neq 0$ is a scalar and $\underline{A}$ and $\underline{C}$ are invertible $\underline{n \times n}$ matrices, then

(1) A^{-1}, and $(A^{-1})^{-1} = A$,

(2) A^k, and $(A^k)^{-1} = (A^{-1})^k$,

(3) A^T, and $(A^T)^{-1} = (A^{-1})^T$,

(4) kA, and $(kA)^{-1} = \frac{1}{k} A^{-1}$,

(5) AC, and $(AC)^{-1} = C^{-1} A^{-1}$, (Note the order)

are all invertible matrices. Moreover, if $\underline{AC}$ is invertible, then so are A and C.

Example:

Simplify $(A^T B)^{-1} A^T$:

$$(A^T B)^{-1} A^T = B^{-1} \underbrace{(A^T)^{-1} A^T}_{I} = B^{-1}.$$

Matrix Inversion Algorithm:

Recall that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then A is invertible $\Leftrightarrow$ $\det(A) \neq 0$,

and

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \det(A) = ad - bc.$$

Goal: Solve $\underline{AB = I_n}$, for the unknown matrix B. (If A is invertible)

(1) write B and I_n as collections of column vectors:

$$B = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n], \quad I_n = [\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n],$$

$\vec{e}_i$ is the ith standard basis of $\mathbb{R}^n$.

(2) Multiply A & B:

$$AB = [A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_n] = [\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n] = I_n,$$

which gives rise to a number of matrix equations to solve,

$$A\vec{v}_1 = \vec{e}_1, \quad A\vec{v}_2 = \vec{e}_2, \ldots, \quad A\vec{v}_n = \vec{e}_n$$

(If A is invertible, then each of these equations will be consistent and will have a unique solution, which are the columns of A^{-1} (or B)).

(3) Row reduce each of the augmented matrices:

$$[A|e_1], [A|e_2], \ldots, [A, e_n].$$

and assuming A is invertible, the RREFs will be as

$$[I|\vec{v}_1], [I, \vec{v}_2], \ldots, [I, \vec{v}_n],$$

which is exactly the row reduction of $[A|\vec{e}_1 \vec{e}_2 \cdots \vec{e}_n] = [A|I]$, whose RREF will be $[I|B] = [I|A^{-1}]$.

Example: $A = \begin{pmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{pmatrix}$. Find A^{-1}?

Apply the elementary row operations to the super-augmented matrix

$$[A \mid I] = \left(\begin{array}{ccc|ccc} 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \underset{R_1 \leftrightarrow R_2}{\sim}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & -1 & 0 & 1 & 0 \\ 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \underset{\substack{-2R_1 + R_2 \to R_2 \\ -R_1 + R_3 \to R_3}}{\sim}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & -1 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{array}\right) \sim \cdots \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{11}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right),$$

Hence, $A^{-1} = \frac{1}{2}\begin{pmatrix} -3 & -3 & 11 \\ 1 & 1 & -3 \\ 1 & -1 & -1 \end{pmatrix}$. (check $AB = BA = I_3$.)

* **Lemma**: If A and B are $n \times n$ matrices such that $AB = I_n$, then $BA = I_n$ as well.

* **Note**: In the matrix inversion algorithm if A does not row reduce to the identity matrix in RREF, or if $\text{rank}(A) < n$, then $A\vec{x} = \vec{b}$ won't have a unique solution, or in particular, A can't be invertible.

Theorem (18.1):

suppose A is an $n\times n$ matrix and $\operatorname{rank}(A)=n$. Then A is invertible and A^{-1} can be computed by the matrix inversion algorithm, or by row reducing

$$[A \mid I] \sim \cdots \sim [I \mid A^{-1}].$$

If $\operatorname{rank}(A) < n$, then A is not invertible.

Examples: Find A^{-1}, if

(1) $A=\begin{pmatrix}1&2&3\\0&0&1\\0&0&4\end{pmatrix}$: $\operatorname{rank}(A)<3 \Rightarrow A$ is not invertible.
(no leading one in the 2^{nd} column)

(2) $A=\begin{pmatrix}1&2&3\\0&1&1\\0&0&1\end{pmatrix}$: $\operatorname{rank}(A)=3 \Rightarrow$

$$[A|I]=\left(\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&1&1&0&1&0\\0&0&1&0&0&1\end{array}\right)\sim\left(\begin{array}{ccc|ccc}1&0&0&1&-2&-3\\0&1&0&0&1&-1\\0&0&1&0&0&1\end{array}\right)$$

$$\Rightarrow A^{-1}=\begin{pmatrix}1&-2&-3\\0&1&-1\\0&0&1\end{pmatrix}$$

(3) $A=\begin{pmatrix}2&2&0\\1&1&2\\1&-1&3\end{pmatrix}$: $[A|I]=\left(\begin{array}{ccc|ccc}2&2&0&1&0&0\\1&1&2&0&1&0\\1&-1&3&0&0&1\end{array}\right)$

$$\sim\left(\begin{array}{ccc|ccc}1&1&2&0&1&0\\2&2&0&1&0&0\\1&-1&3&0&0&1\end{array}\right)\sim\cdots\sim\left(\begin{array}{ccc|ccc}1&0&0&5/8&-3/4&1/2\\0&1&0&-1/8&3/4&-1/2\\0&0&1&-1/4&1/2&0\end{array}\right)$$

$$\Rightarrow A^{-1}=\frac{1}{8}\begin{pmatrix}5&-6&4\\-1&6&-4\\-2&4&0\end{pmatrix}$$

Theorem (18.2): Invertible Matrix Theorem

Let A be a $n \times n$ matrix, then the followings are equivalent:

(1) $\text{rank}(A) = n$,

(2) $A\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$ (trivial solution is the only solution),

(3) $A\vec{x} = \vec{b}$ is consistent for all $b \in \mathbb{R}^n$,

(4) Every linear system $A\vec{x} = \vec{b}$ has a unique solution,

(5) The RREF of A is I_n,

(6) $\text{Null}(A) = \{0\}$,

(7) $\text{Col}(A) = \mathbb{R}^n$,

(8) $\text{Row}(A) = \mathbb{R}^n$,

(9) $\text{rank}(A^T) = n$,

(10) Columns of A are LI,

(11) Rows of A are LI,

(12) Columns of A span $\mathbb{R}^n$,

(13) Rows of A span $\mathbb{R}^n$,

(14) Columns of A form a basis for $\mathbb{R}^n$,

(15) Rows of A form a basis for $\mathbb{R}^n$,

(16) A is invertible,

(17) A^T is invertible.

Examples: Check all the statements for $A_1 = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 3 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.