Chapter 17 : "Bases & Invertible Matrices"

"Conservation of dimension" of Matrix multiplication :

$$\dim(\text{Null}(A)) + \dim(\text{Row}(A)) = \dim(\text{Null}(A)) + \dim(\text{Col}(A)) = n,$$

provided that A is an $m \times n$ matrix, since

$$\dim(\text{Col}(A)) = \dim(\text{Row}(A)) = \text{rank}(A),$$

and we have the "rank-nullity theorem", as

$$\dim(\text{Null}(A)) + \text{rank}(A) = n.$$

Recall that :

- $\text{Null}(A) = \{ \vec{x} \mid A\vec{x} = \vec{0} \}$: a subspace of $\mathbb{R}^n$,
- $\text{Col}(A) = \text{im}(A) = \{ A\vec{x} \mid \vec{x} \in \mathbb{R}^n \}$: a subspace of $\mathbb{R}^m$,
- Row space and column space algorithms can be used to find bases for any subspace W of $\mathbb{R}^n$, provided we start with a spanning set of W.

*But, the fact is that they work for any subspace of any vector space :

Example : W subspace of $\mathbb{P}_3$, and

$$W = \text{span}\{ 3 + x + 4x^2 + 2x^3,\ 2 + 4x + 6x^2 + 8x^3,\ 1 + 3x + 2x^2 + 6x^3,\ -1 + 2x + x^2 + 4x^3 \}$$

(Recall the standard basis of $\mathbb{P}_3 = \{1, x, x^2, x^3\}$)

Re-write the spanning vectors:

$$\vec{u}_1 = \begin{pmatrix} 3 \\ 1 \\ 4 \\ 2 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 2 \\ 4 \\ 6 \\ 8 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 6 \end{pmatrix}, \quad \vec{u}_4 = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 4 \end{pmatrix},$$

$W = \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$. Use the row space algorithm:

$$A = \begin{pmatrix} \vec{u}_1 \\ \vec{u}_2 \\ \vec{u}_3 \\ \vec{u}_4 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 4 & 2 \\ 2 & 4 & 6 & 8 \\ 1 & 3 & 4 & 6 \\ -1 & 2 & 1 & 4 \end{pmatrix} \sim \cdots \sim \hat{A} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which concludes that $\{(1,0,1,0), (0,1,1,2)\}$ is a basis for W, or

$\{1+x^2, x+x^2+2x^3\}$ is a basis for W.

④ Enlarging LI sets to bases:

Example: $\{(1,2,3,1), (1,2,3,2)\}$ is a basis for a subspace of $\mathbb{R}^4$.

Extend it to a basis for $\mathbb{R}^4$:

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 2 & 3 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ REF},$$

in which there are missing pivots in the 2^{nd} & 3^{rd} columns. Therefore, the two vectors that we will add to our set are:

$$(0,1,0,0) \ \& \ (0,0,1,0).$$

Now, check if $\{(1,2,3,1), (1,2,3,2), (0,1,0,0), (0,0,1,0)\}$ spans $\mathbb{R}^4$:

$$B = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 2 & 3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \sim \cdots \sim \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \dim(\text{Row}(B)) = 4$$

rows are basis of $\mathbb{R}^4$.

Note: Using row reduction, we are able to obtain bases for subspaces of $\mathbb{R}^n$. Moreover, by writing vectors in terms of coordinates relative to a standard basis, the techniques also apply for any subspace of any vector space (by identifying vectors with elements of $\mathbb{R}^n$.).

Remark:

(1) Every consistent system has a unique solution $\Leftrightarrow$

the columns of the associated $m \times n$ matrix are LI in $\mathbb{R}^m$.

(2) Every system is consistent $\Leftrightarrow$ its columns span $\mathbb{R}^m$.

Theorem (17.1): A an $n \times n$ square matrix. Then,

(1) $\text{rank}(A) = n$,
(2) $\text{rank}(A^T) = n$,
(3) Every linear system $A\vec{x} = \vec{b}$ has a unique solution,
(4) The RREF of A is I_n,
(5) $\text{Null}(A) = \{0\}$,
(6) $\text{Col}(A) = \mathbb{R}^n$,
(7) $\text{Row}(A) = \mathbb{R}^n$,
(8) Columns of A are linearly independent,
(9) Rows of A are linearly independent,
(10) Columns of A form a basis for $\mathbb{R}^n$,
(11) Rows of A form a basis for $\mathbb{R}^n$.

Matrices of such kind are called "invertible matrices".

Remark : The followings are equivalent for an $m\times n$ matrix A :

(1) $\operatorname{rank}(A) = n$,

(2) The rows of A span $\mathbb{R}^n$,

(3) The columns of A are linearly independent in $\mathbb{R}^m$,

(4) The $n\times n$ matrix A^TA is invertible,

(5) $CA = I_n$ for $n\times m$ matrix C,

(6) If $A\vec{x} = \vec{0}$, $\vec{x}\in\mathbb{R}^n$, then $\vec{x} = \vec{0}$.

Remark : The followings are equivalent for an $m\times n$ matrix A :

(1) $\operatorname{rank}(A) = m$,

(2) The columns of A span $\mathbb{R}^m$,

(3) The rows of A are linearly independent in $\mathbb{R}^n$,

(4) The $m\times m$ matrix AA^T is invertible,

(5) $AC = I_m$ for some $n\times m$ matrix C,

(6) The system $A\vec{x} = \vec{b}$ is consistent for every $\vec{b}\in\mathbb{R}^m$.