# Chapter 16: Bases of Subspaces of $\mathbb{R}^n$

## (Row and column space algorithms).

problems involving:

(1) Given a spanning set $\{\vec{\omega}_1, \cdots \vec{\omega}_m\}$ for a subspace $W$,

(a) finding a basis for $W$,

(b) finding a basis for $W$, which is a subset of $\{\vec{\omega}_1, \cdots, \vec{\omega}_m\}$,

(2) Given an LI set $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in $\mathbb{R}^n$, extend it to a basis of $\mathbb{R}^n$.

- In these problems, we have to find bases for the row & the column spaces of a matrix.

## * The row space algorithm:

In the row reduction procedure, the new rows are linear combinations of the old rows and since the elementary row operations can be reversed, the old rows are linear combinations of the new rows as well.

**Proposition:** (The row space is invariant under row equivalence).

If $A$ is row equivalent to $B$, then $Row(A) = Row(B)$. That is the spans of their rows are exactly the same subspace of $\mathbb{R}^n$.

- Therefore, if $A$ is any matrix, we can carry $A \to B$ by elementary row operations, where $B$ is RREF. Since $Row(A) = Row(B)$, they will have same basis.

Example: $A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & -2 & -8 & 4 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 4 & 1 \end{pmatrix}$

By definition: $\text{Row}(A) = \text{span}\{(1,2,2,3), (2,-2,-8,4), (1,1,0,1), (0,2,4,1)\}$, which we don't know if the spanning set is LI or not.

Instead, let's row reduce $A$ to get $\tilde{A}$, which is RREF;

$$A \underset{\text{(row equivalent)}}{\sim} \tilde{A} = \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\text{Row}(\tilde{A}) = \text{span}\{(1,0,-2,0), (0,1,2,0), (0,0,0,1)\}$, which is LI too, and since $\text{Row}(A) = \text{Row}(\tilde{A})$, we conclude that "non-zero rows of $\tilde{A}$ form a basis for $\text{Row}(A)$".

Ⓐ Theorem (16.1): (The row space algorithm for a basis of $\text{Row}(A)$)

The non-zero rows in any REF of $A$ form a basis for $\text{Row}(A)$, and so $\dim(\text{Row}(A)) = \text{rank}(A)$. For RREF of $A$, we call them the "standard basis" for the subspace $\text{Row}(A)$.

Note: Given a spanning set for a subspace $W$, write a matrix $A$ such that the vectors in $W$ are the rows in $A$, so that $W = \text{Row}(A)$. By applying the row space algorithm we can find a basis for the subspace $W = \text{Row}(A)$.

Example: Find a basis for $U = \text{Span}\{(1,1,2,3),(2,4,1,0),(1,5,-4,-9)\}$.

$$U = \text{Row}(A), \text{ where } A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 4 & 1 & 0 \\ 1 & 5 & -4 & -9 \end{pmatrix} \Rightarrow \text{row reduction}$$

$$\tilde{A} = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -3/2 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \{(1,1,2,3), (0,1,-\tfrac{3}{2},-3)\} \text{ is a basis}$$

for the subspace $U$.

Remark: Given an LI set $\{\vec{u}_1, \dots, \vec{u}_k\}$ in $\mathbb{R}^n$, to extend it to a basis of $\mathbb{R}^n$ (assuming $n > k$), we write an $n \times n$ matrix $A$, whose first $k$ rows are $\vec{u}_1, \dots, \vec{u}_k \in \mathbb{R}^n$ and leave the $n-k$ unknown vectors $\vec{u}_{k+1}, \dots, \vec{u}_n \in \mathbb{R}^n$ as ~~variables~~ symbols, i.e.

$$A = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & & & \vdots \\ u_{k1} & u_{k2} & \cdots & u_{kn} \\ \hline u_{k+1,1} & u_{k+1,2} & \cdots & u_{k+1,n} \\ \vdots & & & \vdots \\ u_{n1} & u_{n2} & \cdots & u_{nn} \end{pmatrix}_{n \times n} \quad \begin{matrix} \left.\right\} = \begin{pmatrix} \vec{u}_1 \\ \vec{u}_2 \\ \vdots \\ \vec{u}_k \end{pmatrix} \subset \mathbb{R}^n \\ \left.\right\} \text{unknowns} \end{matrix}$$

If $\text{rank}(A) = n$, then the rows of $A$ will be a basis of $\mathbb{R}^n$. So, we row reduce the first $k$ rows of $A$ and see which columns the pivots are in.

Then we can choose from the standard basis of $\mathbb{R}^n$ to make sure that there is a pivot in every column of $A$.

Example : Extend $\{(1,2,3,4), (2,4,7,8)\}$ to a basis of $\mathbb{R}^4$.

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 8 \\ & \vec{u}_3 & & \\ & \vec{u}_4 & & \end{pmatrix} \underset{-2R_1+R_2 \to R_2}{\sim} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \\ & \vec{u}_3 & & \\ & \vec{u}_4 & & \end{pmatrix}$$

Therefore, if we set $\vec{u}_3 = (0,1,0,0)$ and $\vec{u}_4 = (0,0,0,1)$, then

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 8 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \underset{R_2 \leftrightarrow R_3}{\sim} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ rank$(A) = 4$ and $\{(1,2,3,4), (2,4,7,8), (0,1,0,0), (0,0,0,1)\}$ is the extension.

## ⊛ Column space algorithm :

Theorem (16.2) : (Basis for Col(A))

Let $A$ be an $m \times n$ matrix. Then a basis for Col(A) consists of those columns of $A$ which give rise to leading ones in an REF of $A$. Hence, $\dim(\text{Col}(A)) = \text{rank}(A)$.

*Remark : suppose $A = [\vec{u}_1 \; \vec{u}_2 \; \cdots \; \vec{u}_n]$ is written in Block column form. To choose a subset of the columns of $A$ which are a basis of $A$, we proceed as the following:

(1) $\{\vec{u}_1\}$ : keep it if it is a non-zero vector.

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(2) $\{\vec{u}_1, \vec{u}_2\}$: If $\vec{u}_2 \in \text{Span}\{\vec{u}_1\}$, then $\{\vec{u}_1, \vec{u}_2\}$ is LD and $\text{span}\{\vec{u}_1, \vec{u}_2\} = \text{span}\{\vec{u}_1\}$, and so discard $\vec{u}_2$. otherwise, keep $\vec{u}_2$.

(3) $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$: If $\vec{u}_3 \in \text{Span}\{\vec{u}_1, \vec{u}_2\}$, then it is redundant and we discard it. otherwise, keep it.

⋮

Continue in this way and throw out $\vec{u}_i$ exactly if the rank of the coefficient matrix of the first i columns of RREF of A equals the rank of the augmented matrix, or exactly when there is <u>no</u> leading one in the ith column of the RREF of A.

<u>Example</u>: $A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & -2 & -8 & 4 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 4 & 1 \end{pmatrix}$ and find a basis for $\text{Col}(A)$.

$A = [\vec{C}_1\ \vec{C}_2\ \vec{C}_3\ \vec{C}_4]$ and $\vec{C}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\vec{C}_2 = \begin{pmatrix} 2 \\ -2 \\ 1 \\ 2 \end{pmatrix}$, $\vec{C}_3 = \begin{pmatrix} 2 \\ -8 \\ 0 \\ 4 \end{pmatrix}$, $\vec{C}_4 = \begin{pmatrix} 3 \\ 4 \\ 1 \\ 1 \end{pmatrix}$

$\{\vec{C}_1\}$ is LI.

$\vec{C}_2 \in \text{span}\{\vec{C}_1\} \Leftrightarrow [\vec{C}_1 | \vec{C}_2] = \left(\begin{array}{c|c} 1 & 2 \\ 2 & -2 \\ 1 & 1 \\ 0 & 2 \end{array}\right)$ is consistent $\Leftrightarrow$

$\text{rank}[\vec{C}_1 | \vec{C}_2] = \text{rank}(\vec{C}_1)$: to check this, let's row reduce $[\vec{C}_1 | \vec{C}_2]$ which will be the first two columns of the RREF of A:

$$\left(\begin{array}{c|c} 1 & 2 \\ 2 & -2 \\ 1 & 1 \\ 0 & 2 \end{array}\right) \sim \left(\begin{array}{c|c} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{array}\right) \text{: inconsistent}$$

That is $\text{rank}([\vec{c}_1 | \vec{c}_2]) > \text{rank}(\vec{c}_1) \Leftrightarrow \vec{c}_2 \notin \text{span}\{\vec{c}_1\} \Leftrightarrow$ $\{\vec{c}_1, \vec{c}_2\}$ is LI.

Now, check if $c_3 \in \text{span}\{\vec{c}_1, \vec{c}_2\} \Leftrightarrow [\vec{c}_1\, \vec{c}_2 | \vec{c}_3]$ is the augmented matrix of a consistent system $\Leftrightarrow \text{rank}([\vec{c}_1\, \vec{c}_2 | \vec{c}_3]) = \text{rank}([\vec{c}_1\, \vec{c}_2])$.

$$[\vec{c}_1\, \vec{c}_2 | \vec{c}_3] = \left(\begin{array}{cc|c} 1 & 2 & 2 \\ 2 & -2 & -8 \\ 1 & 1 & 0 \\ 0 & 2 & 4 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) : \text{consistent}$$

$\Leftrightarrow \text{rank}([\vec{c}_1\, \vec{c}_2 | \vec{c}_3]) = \text{rank}([\vec{c}_1\, \vec{c}_2])$, i.e. there is no leading one in the third column of the RREF of $A \Leftrightarrow \vec{c}_3 \in \text{span}\{\vec{c}_1, \vec{c}_2\}$

So, we discard $\vec{c}_3$, since $\{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ is LD, or

$$\text{span}\{\vec{c}_1, \vec{c}_2\} = \text{span}\{\vec{c}_1, \vec{c}_2, \vec{c}_3\}.$$

Next: $c_4 \in \text{span}\{\vec{c}_1, \vec{c}_2\} = \text{span}\{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ ?

It is true $\Leftrightarrow [\vec{c}_1\, \vec{c}_2\, \vec{c}_3 | \vec{c}_4]$ is consistent

$$\Leftrightarrow \text{rank}([\vec{c}_1\, \vec{c}_2\, \vec{c}_3 | \vec{c}_4]) = \text{rank}([\vec{c}_1\, \vec{c}_2\, \vec{c}_3]):$$

$$[\vec{c}_1\, \vec{c}_2\, \vec{c}_3 | \vec{c}_4] \sim \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) : \text{inconsistent} \Leftrightarrow$$

$\text{rank}([\vec{c}_1\, \vec{c}_2\, \vec{c}_3 | \vec{c}_4]) > \text{rank}([\vec{c}_1\, \vec{c}_2\, \vec{c}_3]) \Leftrightarrow$

$c_4 \notin \text{span}\{\vec{c}_1, \vec{c}_2, \vec{c}_3\} \Leftrightarrow \{\vec{c}_1, \vec{c}_2, \vec{c}_4\}$ is LI &

$\text{span}\{\vec{c}_1, \vec{c}_2, \vec{c}_4\} = \text{span}\{\vec{c}_1, \vec{c}_2, \vec{c}_3, \vec{c}_4\} \to \{\vec{c}_1, \vec{c}_2, \vec{c}_4\}$ is a basis.

**Corollary**: For any matrix $A$:

$$\dim(\mathrm{Row}(A)) = \dim(\mathrm{Col}(A)) = \dim(\mathrm{Col}(A^T)) = \dim(\mathrm{Row}(A^T))$$

$$= \mathrm{rank}(A) = \mathrm{rank}(A^T).$$

No matter what size the matrix is, the maximum number of independent rows $(\dim(\mathrm{Row}(A)))$ is exactly the same as maximum number of independent columns $(\dim(\mathrm{Col}(A)))$.

**Example**:

(1) $$A = \begin{pmatrix} 2 & 1 & 9 & 3 & 7 \\ 3 & -1 & 11 & 1 & 2 \\ 1 & 1 & 5 & 1 & 4 \end{pmatrix} \sim \tilde{A} = \begin{pmatrix} 1 & 0 & 4 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\Rightarrow \mathrm{Col}(A) = \left\{ \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

(2) $$A = \begin{pmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{pmatrix} \sim \tilde{A} = \begin{pmatrix} 1 & 2 & 2 & -1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ columns 1 and 3 of $A$ are a basis of $\mathrm{Col}(A)$, so

$$\mathrm{Col}(A) = \left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \right\}.$$

**Remark**: Given a spanning set for a subspace $U$, from which we wish to extract a basis, write a matrix $A$, whose columns are the given vectors from $U$, so $U = \mathrm{Col}(A)$, and then use the column space algorithm to find a basis for $U$, which contains some of the columns of $A$.

Example: $U = \text{Span}\{(1,2,1,0),(2,-2,1,2),(2,-8,0,4),(3,4,1,1)\}$.

Find a basis of $U$, which is a subset of the given spanning set:

$$A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & -2 & -8 & 4 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 4 & 1 \end{pmatrix} \sim \hat{A} = \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So, columns 1, 2, 4 of $A$ are a basis for $U = \text{Col}(A)$:

$\text{Col}(A) = \{(1,2,1,0),(2,-2,1,2),(3,4,1,1)\}$

*Summary:

There are two diffenent ways to get a basis from a spanning set. For instance for the subspace $W = \text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$:

(1) write $\vec{v}_1, \dots, \vec{v}_k$ as rows of a matrix $B$ and row reduce $B$ to its RREF and take the non-zero rows of RREF, since $W = \text{Row}(B)$.

(2) write $\vec{v}_1, \dots, \vec{v}_k$ as columns of a matrix $A = [\vec{v}_1^T \ \vec{v}_2^T \dots \vec{v}_k^T]$. Row reduce $A$ and take the columns of $A$ which give rise to leading ones in the RREF. In this case, the basis is a subset of the spanning set $\{\vec{v}_1, \dots, \vec{v}_k\}$.

*Remark: $\vec{v} \in \text{Null}(A) \iff \begin{pmatrix} \vec{u}_1 \\ \vec{u}_2 \\ \vdots \\ \vec{u}_n \end{pmatrix} \vec{v} = 0 \iff \begin{pmatrix} \vec{u}_1 \cdot \vec{v} \\ \vec{u}_2 \cdot \vec{v} \\ \vdots \\ \vec{u}_n \cdot \vec{v} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

that is $\vec{v} \in \text{Null}(A) \iff \vec{v}$ is orthogonal to every row of $A$. Also,

$\dim(\text{Null}(A)) + \text{rank}(A) = n \Rightarrow \dim(\text{Null}(A)) + \dim(\text{Row}(A)) = n$.

Example: Find a basis of $Col(A)$, where $A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & -2 & -8 & 4 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 4 & 1 \end{pmatrix}$,

Using $Null(A)$:

$$Col(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -8 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

$$A \sim \tilde{A} = \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} : \text{RREF}$$

$$Null(A) = \{ \vec{v} = (a_1, a_2, a_3, a_4) \mid A\vec{v} = \vec{0} \}$$
$$= \{ (a_1, a_2, a_3, a_4) \mid a_1 \vec{c}_1 + a_2 \vec{c}_2 + a_3 \vec{c}_3 + a_4 \vec{c}_4 = \vec{0} \}$$

Given $\tilde{A}$ in RREF: $Null(A) = \{ (2r, -2r, r, 0) \mid r \in \mathbb{R} \}$

$\Rightarrow (2r)\vec{c}_1 + (-2r)\vec{c}_2 + r\vec{c}_3 + 0\vec{c}_4 = \vec{0}$, for all $r \in \mathbb{R}$. Let $r=1$:

$\Rightarrow \vec{c}_3 = -2\vec{c}_1 + 2\vec{c}_2 \Rightarrow \vec{c}_3 \in \text{span}\{\vec{c}_1, \vec{c}_2\}$

Therefore, $Col(A) = \text{Span}\{\vec{c}_1, \vec{c}_2, \vec{c}_4\}$, but is $\{\vec{c}_1, \vec{c}_2, \vec{c}_4\}$ LI?

Since $\text{Span}\{\vec{c}_1, \vec{c}_2\} = \text{Span}\{\vec{c}_1, \vec{c}_2, \vec{c}_3, \vec{c}_4\}$ $\Rightarrow$ put them as columns of a matrix A & row reduce it, which is the same $\tilde{A}$ as above except the fact that we have to cover up (ignore) the third column (or $\vec{c}_3$) during row reduction. The resulting matrix in RREF (which is the matrix $\tilde{A}$, without the third column) contains a leading one on every column and thus $\{\vec{c}_1, \vec{c}_2, \vec{c}_4\}$ is LI and so a basis for $Col(A)$.