Chapter 15: Vector Spaces Associated with Matrices

* Column space, Row space and Nullspace:

Let A be an $m \times n$ matrix, then

<u>Definition</u>: The column space of A (also called the "image of A", denoted also by $\operatorname{im}(A)$) is

$$\operatorname{Col}(A) = \operatorname{Span}\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}.$$

where $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n$ are <u>$m \times 1$</u> columns of A (vectors in $\mathbb{R}^m$).

Recall that if $\vec{x} \in \mathbb{R}^n$, then $A\vec{x}$ is a linear combination of the columns of A. So,

$$\operatorname{Col}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$$

* Since $\operatorname{Col}(A)$ is given as the span of some vectors in $\mathbb{R}^m$, it is a subspace of $\mathbb{R}^m$.

<u>Definition</u>: The "row space" of A is:

$$\operatorname{Row}(A) = \operatorname{span}\{\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m\},$$

where $\{\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m\}$ are the rows of A (vectors in $\mathbb{R}^n$).

* $\operatorname{Row}(A)$ is a subspace of $\mathbb{R}^n$.

<u>Definition</u> : The "nullspace of A" (also known as "kernel" of A, also denoted by Ker(A)) is

$$Null(A) = \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0} \}.$$

or, the nullspace is the general solution to the homogeneous linear system given by $A\vec{x} = \vec{0}$.

<u>Lemma</u> : Nullspace is a subspace of $\mathbb{R}^n$.

<u>Proof</u> : Size A = m x n (for a linear system with m equations and n variables, meaning that it consists of vectors with n-components. Hence, $Null(A) \subset \mathbb{R}^n$.

For the rest of the proof, we can apply the subspace test. $\square$

⊛ <u>Note</u> : these vector spaces, eg. Col(A), Row(A), Null(A), are usually distinct from one another.

* Finding basis of Null(A) :

<u>Example</u> :
$$A = \begin{pmatrix} 1 & 2 & 3 & -3 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

Null(A) is the general solution to the homogeneous equation $A\vec{x} = \vec{0}$. So, the first step is to solve that linear system

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & -3 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{array}\right) \underset{\substack{-R_1+R_3 \\ \to R_3}}{\sim} \left(\begin{array}{cccc|c} 1 & 2 & 3 & -3 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & -2 & -2 & 4 & 0 \end{array}\right)$$

$$\sim \quad \begin{array}{l} -2R_2+R_1 \to R_1 \\ 2R_2+R_3 \to R_3 \end{array} \quad \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \text{(RREF)}$$

Set the non-leading variables equal to parameters ($x_3 = r$, $x_4 = t$). and solve for leading variables in terms of parameters, and finally

$$x_1 = -r-t, \quad x_2 = -r+2t, \quad x_3 = r, \quad x_4 = t$$

$$\text{Null}(A) = \left\{ \begin{pmatrix} -r-t \\ -r+2t \\ r \\ t \end{pmatrix} \,\middle|\, t, r \in \mathbb{R} \right\}$$

$$= \left\{ r \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \,\middle|\, t, r \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Since, these vectors span Null(A) and are LI (not being multiple of each other), then $\{(-1,-1,1,0), (-1,2,0,1)\}$ is a basis for Null(A)

<u>Theorem (15.1)</u>: (Basic solutions form a basis for Null(A))

The spanning set of Null(A) obtained from the RREF form of $[A \mid 0]$ (or, the set of basic solutions of $A\vec{x} = \vec{0}$) is a basis for Null(A).

<u>Proof</u>: please, refer to the primary text.

<u>Corollary (rank-Nullity theorem)</u>

The dimension of the nullspace of A is equal to the number of non-leading variables of A: $\boxed{\dim(\text{Null}(A)) + \text{rank}(A) = n \ (\#\text{ Columns of } A)}$

Solving Inhomogeneous Systems:

Example: Solve $A\vec{x} = \vec{b}$, where $A = \begin{pmatrix} 1 & 2 & 3 & -3 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 10 \\ 3 \\ 4 \end{pmatrix}$

Row reducing the augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & -3 & 10 \\ 0 & 1 & 1 & -2 & 3 \\ 1 & 0 & 1 & 1 & 4 \end{array}\right) \underset{\substack{-R_1+R_3 \\ \to R_3}}{\sim} \left(\begin{array}{cccc|c} 1 & 2 & 3 & -3 & 10 \\ 0 & 1 & 1 & -2 & 3 \\ 0 & -2 & -2 & 4 & -6 \end{array}\right)$$

$$\underset{\substack{2R_2+R_3 \\ \to R_3}}{\overset{-2R_2+R_1 \to R_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \text{ : RREF, so}$$

$$\left\{ \begin{pmatrix} 4-r-t \\ 3-r+2t \\ r \\ t \end{pmatrix} \middle| \; t, r \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 4 \\ 3 \\ 0 \\ 0 \end{pmatrix} + r\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \middle| \; r, t \in \mathbb{R} \right\}$$

The general solution of the inhomogeneous system =

A particular solution of inhomogeneous system + general solution of hom. system;

(or, translate the Null(A) from the origin).

$\underbrace{\text{translate}}_{\text{any particular solution of inhom. system.}}$

***Theorem (15.2):** "Inhomogeneous Systems & Nullspace"

Suppose $A\vec{x} = \vec{b}$ is a consistent linear system. Then,

(next page).

(1) If $\vec{x} = \vec{v}$ is a solution to the system $A\vec{x} = \vec{b}$ and $\vec{x} = \vec{u}$ is any solution to the homogeneous system $A\vec{x} = \vec{0}$, then $\vec{u} + \vec{v}$ is a solution to $A\vec{x} = \vec{b}$.

(2) If $\vec{w}$ & $\vec{v}$ are two solutions to $A\vec{x} = \vec{b}$, then $\vec{x} = \vec{w} - \vec{v}$ is a solution to $A\vec{x} = \vec{0}$.

proof:

(1) $A\vec{v} = \vec{b}$, $A\vec{u} = \vec{0} \Rightarrow A(\vec{v} + \vec{u}) = A\vec{v} + A\vec{u} = \vec{b} + \vec{0} = \vec{b}$,

(2) $A\vec{v} = \vec{b}$, $A\vec{w} = \vec{b} \Rightarrow A(\vec{w} - \vec{v}) = A\vec{w} - A\vec{v} = \vec{b} - \vec{b} = \vec{0}$. $\square$

Notes:

If $A\vec{x} = \vec{0}$ has a unique solution, so does any consistent $A\vec{x} = \vec{b}$, and if a homogeneous system $A\vec{x} = \vec{0}$ has infinitely many solutions (indexed by k basic solutions), then any consistent $A\vec{x} = \vec{b}$ will also have k parameters in its solutions.

If $\vec{b} \notin \mathrm{Col}(A)$, there won't be a solution to $A\vec{x} = \vec{b}$ at all.

*Summary: (Consistency of linear systems) $\boxed{A: m \times n \text{ matrix}}$

A linear system is consistent if & only if:

(1) $\vec{b}$ is a linear combination of the columns of A, iff

(2) $\vec{b} \in \mathrm{Col}(A)$, iff

(3) $\mathrm{rank}(A) = \mathrm{rank}([A \mid b])$.

Theorem (15.3):

The equation $A\vec{x}=\vec{b}$ will be consistent for all $\vec{b} \in \mathbb{R}^m$, iff:

(1) there are no zero rows in RREF of A, iff
(2) every $\vec{b} \in \mathbb{R}^m$ is a linear combination of the columns of A, iff
(3) $Col(A) = \mathbb{R}^m$, iff.
(4) $\dim(Col(A)) = m$, iff
(5) $rank(A) = m$.

* Summary: Number of solutions of a consistent system

A consistent system $A\vec{x}=\vec{b}$ has a unique solution iff

(1) every variable is a leading variable, iff
(2) there is a leading 1 in every column of the RREF of A, iff
(3) $A\vec{x}=\vec{0}$ has a unique solution, iff
(4) the columns of A are LI, iff
(5) $Null(A) = \{\vec{0}\}$, iff
(6) $\dim(Null(A)) = 0$, iff
(7) $rank(A) = n$.

Example:

Subspace of $\mathbb{R}^n$: (1) $W = Null(A)$, for some matrix A.
(2) $W = Span\{\vec{v}_1, \dots, \vec{v}_m\}$.

for instance, let $W = \text{span}\{(1,1,2)\}$ and find a matrix A, such that $W = \text{Null}(A)$:

$(x,y,z) \in W$ iff $\left[\begin{array}{c|c} 1 & x \\ 1 & y \\ 2 & z \end{array}\right]$ is consistent. After row reduction,

$\left(\begin{array}{c|c} 1 & x \\ 0 & y-x \\ 0 & z-2x \end{array}\right)$, which is consistent iff $y-x = z-2x = 0$, so,

$$W = \{(x,y,z) \in \mathbb{R}^3 \mid \underbrace{x-y = z-2x = 0}_{\text{intersection of two planes.}}\} \Rightarrow A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$$

$\Rightarrow W = \text{Null}(A)$.

__Example__: $W = \text{span}\{(1,0,0,1), (1,1,1,0), (2,1,-1,1)\}$:

$W = \text{Col}(A)$, where $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$. Therefore,

$(x,y,z,w) \in W$, iff the linear system with augmented matrix

$\left(\begin{array}{ccc|c} 1 & 1 & 2 & x \\ 0 & 1 & 1 & y \\ 0 & 1 & -1 & z \\ 1 & 0 & 1 & w \end{array}\right)$ is consistent. After row reduction;

$\left(\begin{array}{ccc|c} 1 & 1 & 2 & x \\ 0 & 1 & 1 & y \\ 0 & 0 & -2 & z-y \\ 0 & 0 & 0 & -x+y+w \end{array}\right)$, which is consistent iff $x-y-w=0$, i.e.

$$W = \{(x,y,z,w) \in \mathbb{R}^4 \mid x-y-w=0\} \Rightarrow W = \text{Null}(\begin{bmatrix} 1 & -1 & 0 & -1 \end{bmatrix}).$$