Chapter 13 : Applications & examples of solving linear systems

Recall from the last chapter that the rank of a matrix A ($\text{rank}(A)$), is the number of leading ones (pivots) in any REF of A.

* The rank doesn't change when we do elementary row operations, and it can never exceed the number of rows or columns of the matrix.

* we usually denote an augmented matrix by $[A \mid \underline{b}]$, where A is the coefficient matrix anb $\underline{b}$ is the constant part of the linear system.

* We can determine if a system is consistent or inconsistent by comparing the rank(A) with rank$([A \mid b])$:

  - suppose we reduce A in $[A \mid b]$ to RREF, then:

(a) $$\left(\begin{array}{ccc|c} * & \cdots & * & * \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{array}\right) \longrightarrow$$ rows with leading ones in the coefficient matrix A. (consistent), or

(b) $$\left(\begin{array}{ccc|c} * & \cdots & * & * \\ 0 & \cdots & 0 & a_1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & a_k \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{array}\right)$$, where $a_i \neq 0$ (inconsistent)

In case (a): $\text{rank}(A) = \text{rank}([A \mid b])$, since all the leading ones occured in the coefficient matrix.

In case (b): $\text{rank}([A \mid b]) = \text{rank}(A) + 1$, since the non-zero entry a_1 will lead to a single new leading $\underline{1}$.

Therefore,

$$\boxed{\operatorname{rank}(A) \leq \operatorname{rank}([A|b]) \leq \operatorname{rank}(A)+1}$$

Example: In homogeneous systems $\operatorname{rank}(A) = \operatorname{rank}([A|0])$. So, the system is consistent.

Summary: $[A|b]$ is the augmented matrix of a linear system.

(1) The system is inconsistent $\Leftrightarrow \operatorname{rank}(A) < \operatorname{rank}([A|b])$.

(2) The system has a unique solution $\Leftrightarrow \operatorname{rank}(A) = \operatorname{rank}([A|b])$ and $\operatorname{rank}(A) = \#$ columns of A.

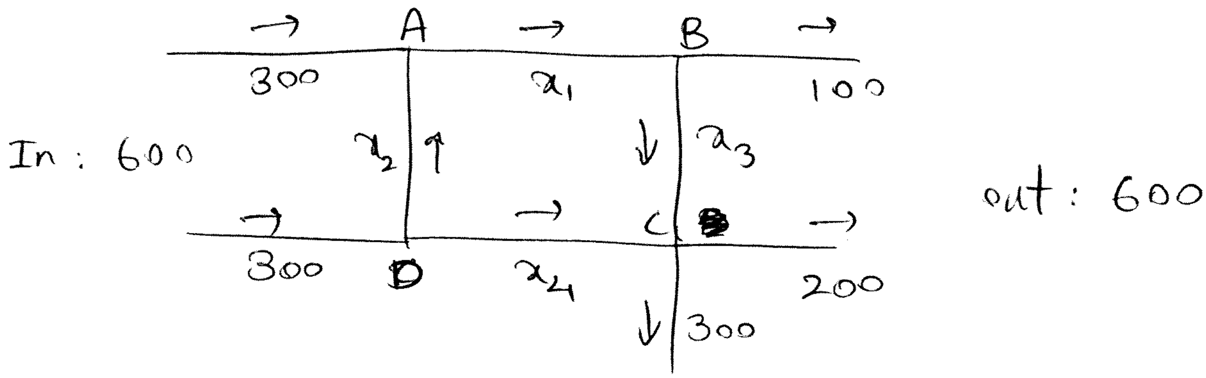
(3) The system has infinitely many solutions $\Leftrightarrow \operatorname{rank}(A) = \operatorname{rank}([A|b])$ and $\operatorname{rank}(A) < \#$ columns of A.

⊕ Applications:

(1) Network and traffic flow problems

Model the internal flow of a network by just understanding its inputs and outputs and how the traffic is restricted in between. We don't expect a unique solution.

Example:



Intersection	Flow in	=	Flow out
A	$300 + x_2$	=	x_1
B	x_1	=	$x_3 + 100$
C	$x_3 + x_4$	=	$200 + 300$
D	300	=	$x_2 + x_4$

Therefore, the linear system is:

$$\begin{aligned} x_1 - x_2 &= 300 \\ x_1 - x_3 &= 100 \\ x_3 + x_4 &= 500 \\ x_2 + x_4 &= 300 \end{aligned} \quad , \text{or} \quad \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 300 \\ 1 & 0 & -1 & 0 & 100 \\ 0 & 0 & 1 & 1 & 500 \\ 0 & 1 & 0 & 1 & 300 \end{array}\right)$$

which will reduce to

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 600 \\ 0 & 1 & 0 & 1 & 300 \\ 0 & 0 & 1 & 1 & 500 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \text{Solution} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 600 - s \\ 300 - s \\ 500 - s \\ s \end{pmatrix},$$

$s \in \mathbb{R}$.

Since $x_i \geq 0$, then $0 \leq s \leq 300$.

Application (II): Testing Scenarios

Find all possible values of k, such that the following system has (a) no solution, (b) a unique solution and (c) infinitely many solutions

$$\begin{aligned} kx+y+z&=1\\ x+ky+z&=1\\ x+y+kz&=1 \end{aligned} \Rightarrow [A|b]=\left(\begin{array}{ccc|c} k & 1 & 1 & 1\\ 1 & k & 1 & 1\\ 1 & 1 & k & 1 \end{array}\right)$$

$$\underset{R_1 \leftrightarrow R_2}{\sim} \left(\begin{array}{ccc|c} 1 & k & 1 & 1\\ k & 1 & 1 & 1\\ 1 & 1 & k & 1 \end{array}\right) \underset{\substack{-kR_1+R_2\to R_2\\ -R_1+R_3\to R_3}}{\sim} \left(\begin{array}{ccc|c} 1 & k & 1 & 1\\ 0 & 1-k^2 & 1-k & 1-k\\ 0 & 1-k & k-1 & 0 \end{array}\right)$$

Two cases:

(1) $k=1 \Rightarrow$ our matrix is $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{array}\right)$, which is RREF and has infinitely many solutions ((c) – above).

(2) $k\neq 1 \Rightarrow k-1\neq 0$ and:

$$\underset{\substack{\frac{1}{1-k}R_2\to R_2\\ \frac{1}{1-k}R_3\to R_3}}{\sim} \left(\begin{array}{ccc|c} 1 & k & 1 & 1\\ 0 & 1+k & 1 & 1\\ 0 & 1 & -1 & 0 \end{array}\right) \underset{R_2\leftrightarrow R_3}{\sim} \left(\begin{array}{ccc|c} 1 & k & 1 & 1\\ 0 & 1 & -1 & 0\\ 0 & 1+k & 1 & 1 \end{array}\right)$$

$$\underset{-(k+1)R_2+R_3\to R_3}{\sim} \left(\begin{array}{ccc|c} 1 & k & 1 & 1\\ 0 & 1 & -1 & 0\\ 0 & 0 & 2+k & 1 \end{array}\right)$$

Here, if $k=-2$, then the system is inconsistent ((a)-above), otherwise, we can divide R_3 by $2+k$ to get a leading one in the third column and it implies that we have a unique solution ((b)-above)

Application (III): Solving vector equations

Example: Does the set $\{(1,2,3),(4,5,6),(7,8,9)\}$ span $\mathbb{R}^3$?

or, $a_1(1,2,3)+a_2(4,5,6)+a_3(7,8,9)\overset{?}{=}(x,y,z)\in\mathbb{R}^3$,

or,

$$\begin{aligned} a_1+4a_2+7a_3&=x \\ 2a_1+5a_2+8a_3&=y \\ 3a_1+6a_2+9a_3&=z \end{aligned} \Rightarrow \left(\begin{array}{ccc|c} 1 & 4 & 7 & x \\ 2 & 5 & 8 & y \\ 3 & 6 & 9 & z \end{array}\right)$$

after row reduction:

$$\left(\begin{array}{ccc|c} 1 & 4 & 7 & x \\ 0 & 1 & 2 & -(y-2x)/3 \\ 0 & 0 & 0 & x-2y+z \end{array}\right),$$

so, the linear system is consistent if and only if $x-2y+z=0$, which is the equation of the plane they span. These three vectors are linearly dependent.