Chapter 12: Solving Linear Systems (continued.)

Last time we introduced the notion of the "augmented" matrix of a linear system and defined the three "elementary row operations":

(1) Add a multiple of one row to another row,

(2) Interchange two rows,

(3) Multiply a row by a non-zero scalar

Definition: two linear systems are equivalent if they have the same general solution.

Theorem (12.1): If an elementary row operation is performed on the augmented matrix of a linear system, the resulting linear system is equivalent to the original one.

Definition: Two matrices A & B are row-equivalent ($A \sim B$ or $B \sim A$) if B can be obtained from A by a finite sequence of elementary row operations.

Goal: Row reduce any linear system to RREF and read the general solution from RREF.

Recall: A matrix (augmented or not) is in row echelon form (REF), if:

(1) All zero rows are at the bottom,

(2) The first non-zero entry in each row is a 1 (leading one or pivot),

(3) Each pivot 1 is to the right of the leading 1s in the rows above.

If, in addition, the matrix satisfies

(4) Each leading 1 is the only non-zero entry in its column,

then the matrix is said to be in reduced row echelon form (RREF).

<u>Theorem (12.2)</u> : (Uniqueness of the RREF)

Every matrix is row equivalent to a "unique" matrix in RREF.

(Matrices in REF are not unique).

* Reading the solution from RREF:

<u>Examples</u>:

(1) The solution to the linear system given by the following RREF (after row reduction):

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right),$$

is unique and given by:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

(2) Suppose we have row reduced and we got the following REF:

$$\left(\begin{array}{cccc|c} 1 & a & 0 & b & d \\ 0 & 0 & 1 & e & f \\ 0 & 0 & 0 & 0 & g \end{array}\right),$$

Then, there are two cases:

(I) If $g \neq 0$, the last row corresponds to a degenerate equation, and so the system is inconsistent.

(II) If $g = 0$, then this system is in RREF and to get the general solution we set non-leading variables (the variables corresponding to columns of the coefficient matrix, which do not have a leading 1) equal to parameters. In this case, $x_2 = s$, $x_4 = t$, and

$$x_1 + as + bt = d, \quad x_2 = s, \quad x_3 + et = f, \quad x_4 = t.$$

or, the general solution is

$$\{(-as - bt + d, s, f - et, t) \mid s, t \in \mathbb{R}\}.$$

<u>Note</u>: Having infinitely many solutions is related to having non-leading variables, not to the non-zero rows.

(3) Suppose RREF is

$$\left(\begin{array}{cccc|c} 1 & a & 0 & 0 & c \\ 0 & 0 & 1 & 0 & d \\ 0 & 0 & 0 & 1 & e \end{array}\right),$$

then we only have one non-leading variable $x_2 = s$, and so

$$\{(c - as, s, d, e) \mid s \in \mathbb{R}\}.$$

is the general solution.

* General rules for reading off the "type" of general solution*
from the REF:

(1) If the system contains a degenerate equation, then it is inconsistent.
So, if the augmented matrix contains a row like

$$[0 \ 0 \cdots 0 \ ; \ b],$$

with $b \neq 0$, then it is inconsistent and the general solution is the empty set.

(2) otherwise,

(I) If every column has a leading 1, then there is a unique solution.

(II) If there is a column which does not have a leading 1, then there are infinitely many solutions.

* General rule for writing down the general solution of a consistent system from the RREF:

(1) If there is a unique solution, then this is the vector in the augmented column.

(2) otherwise, identify leading & non-leading variables:

(I) Each leading variable corresponds to one row of the augmented matrix; write down the equation for this row and solve for the leading variable in terms of the non-leading variables.

(II) Set each non-leading variable equal to a different parameter.

(III) write down the general solution, including all variables.

Example: suppose the RREF of our system is

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right),$$

then, the system is:

$$x_1 + 2x_4 = 3, \quad x_3 + x_4 = 4, \quad x_5 = 0,$$

which is consistent. The leading variables are x_1, x_3, x_5 and the non-leading variables are x_2, x_4. So, we set

$$x_2 = s, \quad x_4 = t,$$

and conclude that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3-2x_4 \\ x_2 \\ 4-x_4 \\ x_4 \\ 0 \end{pmatrix} = \begin{pmatrix} 3-2t \\ s \\ 4-t \\ t \\ 0 \end{pmatrix},$$

so, the general solution in parametric vector form is:

$$\left\{ \begin{pmatrix} 3 \\ 0 \\ 4 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \;\middle|\; t, s \in \mathbb{R} \right\}$$

* Reducing systems to REF & RREF: Gaussian Elimination

The Gaussian eliminitian algorithm: It can be applied to any matrix C, and stops at a matrix $\mathcal{E}$, which is RREF.

Steps:

(1) If C is zero matrix, stop.

(2) Find the left-most non-zero column, and interchange the top row with another, if necessary to bring a non-zero entry to the first row of this column.

(3) Scale the first row, as necessary, to get a leading 1.

(4) Annihilate the rest of the column below using this leading 1 as a pivot. That is, if a_i is the entry in this column of row i, then add $-a_i R_1$ to R_i and put the result back in the ith row.

(5) This completes the operations with the first row. Now, keep first row and go back to step (1).

When the algorithm stops, the resulting matrix will be in REF. Now proceed with the following steps to put the matrix in RREF:

(6) If the right-most leading 1 is in row 1, stop.

(7) Start with the right-most leading 1, which will be in the last non-zero row. Use it to annihilate every entry above it in its column. That is if $a_i \neq 0$ is the entry in this column in row i, then add $-a_i$ times this row to R_i and put the result back in the ith row.

(8) cover up the row you used and go to step (6).

Example: Let's run the Gaussian elimination algorithm on

$$C = \begin{pmatrix} 0 & 0 & +2 & -2 \\ 1 & 1 & 3 & -1 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{pmatrix}.$$

(1) $C \neq$ zero matrix, so proceed.

(2) interchange rows 1 & 2:

leading 1 (Pivot)

$$\begin{pmatrix} 0 & 0 & +2 & -2 \\ 1 & 1 & 3 & -1 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{pmatrix} \overset{R_1 \leftrightarrow R_2}{\sim} \begin{pmatrix} 1 & 1 & 3 & -1 \\ 0 & 0 & +2 & -2 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{pmatrix}$$

(3) No need to rescale: we already have a leading 1 in row 1.

(4) subtract the first row from rows 3 & 4:

$$\begin{pmatrix} 1 & 1 & 3 & -1 \\ 0 & 0 & 2 & -2 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{pmatrix} \underset{-R_1+R_4 \to R_4}{\overset{-R_1+R_3 \to R_3}{\sim}} \begin{pmatrix} 1 & 1 & 3 & -1 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -3 & 3 \end{pmatrix}$$

(5) ignore the first row & go to step (1).

(1) The remaining matrix is not zero matrix, so proceed.

(2) now, the left-most non-zero column is column 3, whose second row entry is 2 (which is the first row of the matrix left, when we ignore the first row in step (5)).

(3) Divide row 2 by 2 to get a leading 1 in the second row:

$$\begin{pmatrix} 1 & 1 & 3 & -1 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -3 & 3 \end{pmatrix} \underset{\sim}{\tfrac{1}{2}R_2 \to R_2} \begin{pmatrix} 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -3 & 3 \end{pmatrix}$$

(4) Clear column three below the leading 1;

$$\begin{matrix} R_2 + R_3 \to R_3 \\ \sim \\ 3R_2 + R_4 \to R_4 \end{matrix} \begin{pmatrix} 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(5) Ignore the first two rows and go back to step (1).

(1) The remaining matrix is zero, so proceed to (6).

(6) The right-most leading 1 is in row two, so proceed to (7).

(7) Clear row one, column 3 (above the leading 1 in row 2):

$$\begin{pmatrix} 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underset{\sim}{-3R_2 + R_1 \to R_1} \begin{pmatrix} 1 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(8) Cover up row 2 and go back to step (6).

(6) Ignore row (2). Now, the right-most leading 1 is in row 1 ! Stop.
(The resulting matrix is now in RREF).

⊛ Using the Gaussian algorithm to solve a linear system:
Apply the row operations to the rows of the whole "augmented matrix", until the "coefficient matrix" is in RREF.

Once the coefficient matrix is in RREF, the general solution can be found as follows:

(1) Decide if the system is consistent or not. If consistent:

(2) Assign parameters to non-leading variables,

(3) Solve for leading variables in terms of parameters.

Recall: The goal is to get the coefficient matrix into RREF, by applying row operations to the rows of the whole augmented matrix. All decision in the algorithm will only depend on the coefficient matrix.

Example: Augmented matrix:

$$\left(\begin{array}{cccc|c} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \end{array}\right)$$

Steps:

(1) not the zero matrix!

(2) The top row has a zero, which cannot be a pivot. so, swap R_1 and R_2:

$$\xrightarrow{R_1 \leftrightarrow R_2} \sim \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 & 6 \end{array}\right)$$

(3) There is a pivot in row 1. Move on.

(4) clear $\underline{2}$ in row three:

$$\underset{-2R_1 + R_3 \to R_3}{\sim} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 & \underline{4} \end{array}\right)$$

(5) Back to (1), and consider R_2 & R_3.

(1) Not the zero matrix. Move on.

(2) The first non-zero column is in column 2. No interchanges neede

(3) It is a leading 1.

(4) Clear (-1) in 3rd row below the leading 1.

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 & -4 \end{array}\right) \underset{R_2+R_3 \to R_3}{\sim} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

(5) Back to (1) and only consider R_3 ~~3rd~~.

(1) It is zero matrix. proceed to (6). The system is consistent and there are non-leading variables, So the system has infinitely many solutions.

(6) The right-most leading 1 is not in row 1. move on.

(7) The right-most leading 1 is in column 2. Annihilate the 2 above that:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \overset{-2R_2+R_1 \to R_1}{\sim} \left(\begin{array}{cccc|c} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

(8) Cover up the second row and go to step 6.

(6) The right-most leading 1 is in row 1. stop.

Now, the coefficient matrix is in RREF.

The non-leading variables are x_3 & x_4, so let $x_3 = s$, $x_4 = t$.

Then, $x_1 = x_3 + 2x_4 - 3 = -3 + s + 2t$, and

$$x_2 = 4 - 2x_3 - 3x_4 = 4 - 2s - 3t,$$

Thus, the general solution is

$$\begin{cases} x_1 = -3 + s + 2t \\ x_2 = 4 - 2s - 3t \\ x_3 = s \\ x_4 = t \end{cases}, \quad t, s \in \mathbb{R}.$$

* Recall that the RREF of a matrix exists and is unique:

<u>Definition</u>: The "rank" of a matrix A, denoted by $\operatorname{rank}(A)$, is the number of leading ones (pivots) in any REF of A.

<u>Example</u>: Rank of $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \end{pmatrix}$ is $\underline{2}$.