Chapter 11: Solving systems of linear equations.

In general, a "linear system" is a collection of m linear equations, in n variables, and the goal is to solve the system "simultaneously".

Example: $\begin{cases} x_1 + 2x_2 + x_4 = 1 \\ x_3 - x_4 = 1 \end{cases}$, $m=2$, $n=4$

* The "general solution" to a linear system is the set of "all" solutions.

Definition: - A linear system that has no solutions is called "inconsistent".

- A linear system that has at least one solution is called "Consistent".
- A linear system in which all the constants on the right hand side are zeros is called "homogeneous".
- A linear system in which at least one of the constants on the right hand side is non-zero is called "inhomogeneous".

Note: Homogeneous linear systems are always consistent, since $\vec{0} = (0, 0, \cdots, 0)$ is always a solution (called trivial solution)

Definition: - A linear equation in which all the coefficients are zero is called "degenerate", it looks like

$$0x_1 + 0x_2 + \cdots + 0x_n = b, \quad b \in \mathbb{R}$$

If $b \neq 0$, this equation has no solution. If $b = 0$, this equation has every vector in $\mathbb{R}^n$ as a solution.

<u>Note</u>: Any linear system containing a degenerate inhomogeneous equation is inconsistent.

<u>Theorem (11.1)</u>: (Types of general solutions) Any linear system with real (or complex) coefficients has either:

1. a unique solution,
2. no solution, or
3. infinitely many solutions.

<u>Solving linear systems</u>:

Example:

$$\left.\begin{aligned} E_1 &: x + y + 2z = 3 \\ E_2 &: x - y + z = 2 \\ E_3 &: y - z = 1 \end{aligned}\right\}$$

1^{st} step: eliminate x by adding $(-1)E_1$ to E_2,

$$\begin{aligned} E_1 &: x + 2y + 2z = 3 \\ -E_1 + E_2 &: -2y - z = -1 \\ E_3 &: y - z = 1 \end{aligned}$$

2^{nd} step: swap the new second and third equations.

$$E_1:\ x + y + 2z = 3$$
$$E_2':\ y - z = 1$$
$$E_3':\ -2y - z = -1$$

3rd step; Add $2E_2'$ to E_3':

$$E_1:\ x + y + 2z = 3$$
$$E_2':\ y - z = 1$$
$$2E_2' + E_3':\ -3z = 1$$

last step: multiply the last equation by $-1/3$:

$$x + y + 2z = 3$$
$$y - z = 1$$
$$z = -1/3$$

we can plug $z = -1/3$ to second equation and find $y = 2/3$ and then plug them to the first equation and find $x = 3$.

* Type of operations we did:

(1) Add a multiple of one row to another row,

(2) Interchange two rows,

(3) Multiply a row by a non-zero scalar.

These are called "elementary row operations" and they are exactly the three operations that will not change the general solutions.

Now, let's replace the linear system by its "augmented matrix":

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

$\underbrace{\qquad\qquad\qquad}_{\text{coefficient matrix}}$

If there were $\underline{m}$ equations and $\underline{n}$ unknowns, then the coefficient matrix is of size $m \times n$. Each column of the coefficient matrix corresponds to one variable of the linear system (e.g. x, y, z).

- Let's perform the elementary row operations for the previous example, but this time to the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & 1 \end{array}\right) \xrightarrow[-R_1+R_2 \to R_2]{} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & -2 & -1 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow[R_2 \leftrightarrow R_3]{} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & -2 & -1 & -1 \end{array}\right) \xrightarrow[2R_2+R_3 \to R_3]{} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -3 & 1 \end{array}\right)$$

$$\xrightarrow[-\frac{1}{3}R_3 \to R_3]{} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1/3 \end{array}\right)$$: Row Echelon Form (REF)

* Row Echelon Form (REF) & Reduced Row Echelon Form (RREF):

Definition: A matrix (augmented or not) is in Row Echelon Form (REF) if:

(1) All zero rows are at the bottom.

(2) The first non-zero entry in each row is a 1 (called a leading one or a pivot).

(3) Each leading one is to the right of the leading 1s in the rows above.

If, in addition, the matrix satisfies:

(4) Each leading 1 is the only non-zero entry in its column,

then the matrix is said to be in reduced row echelon form (RREF).

Examples:

(1) $\left(\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 5 \\ 0 & 0 & 1 & 1 & 2 & 0 \end{array}\right)$ is in REF, but not RREF.

(2) $\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$ is also in REF, but not RREF.

(3) suppose we have the following RREF: $\left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right)$

Then, the corresponding linear system is: $x=a$, $y=b$, $z=c$.

<u>Note</u>: From REF we will be able to tell if the system is inconsistent, or has a unique solution, or has infinitely many solutions. From RREF we can read off the solution!

(4) suppose we have:

$$\left(\begin{array}{cccc|c} 1 & a & 0 & b & d \\ 0 & 0 & 1 & e & f \\ 0 & 0 & 0 & 0 & g \end{array}\right)$$

I) If $g \neq 0 \Rightarrow$ degenerate equation & the system is inconsistent.

II) If $g=0 \Rightarrow$ the system is in RREF and we set the non-leading variables (variables corresponding to columns which do not have a leading 1) to parameters. Here, set $x_2 = s$, $x_4 = t$, then

$x_1 = -as - bt + d$, $x_2 = s$, $x_3 = f - et$, $x_4 = t$.

general solution: $\{(-as-bt+d, s, f-et, t) \mid s,t \in \mathbb{R}\}$.

(5) Suppose our RREF is

$$\left(\begin{array}{cccc|c} 1 & a & 0 & 0 & c \\ 0 & 0 & 1 & 0 & d \\ 0 & 0 & 0 & 1 & e \end{array}\right)$$

Then, we only have one non-leading variable $x_2 = s$, so

general solution: $\{(c-as, s, d, e) \mid s \in \mathbb{R}\}$.