Chapter 8 : Linearly Independence and Spanning sets.

Last time, we saw that a set of vectors $\{\vec{v}_1, \dots, \vec{v}_m\}$ is "linearly independent" (LI) if the only solution to the <u>dependence equation</u>

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_m\vec{v}_m = \vec{0}$$

is the trivial solution ($a_1 = 0, a_2 = 0, \dots, a_m = 0$), and the set of vectors is "linearly dependent" (LD) if there is a non-trivial solution to

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_m\vec{v}_m = \vec{0}$$

(i.e. a solution in which not coefficients are zeros).

<u>Facts about LI & LD :</u>

1) A set $\{\vec{v}\}$ consisting of just one vector is LI if and only if $\vec{v} \neq \vec{0}$.
2) If a set S is LD, then any set containing S is also LD.
3) If a set S is LI, then any subset of S is also LI.
4) $\{\vec{0}\}$ is LD.
5) Any set containing the zero vector $\vec{0}$ is LD.
6) A set with two vectors is LD if and only if one of the vectors is a multiple of the other.
7) A set with three or more vectors could be LD even if no two vectors are multiples of one another.

Theorem (8.1): Relation between LI & spanning

A set $\{\vec{v}_1, \dots, \vec{v}_m\}$ is LD if and only if there is at least one vector $\vec{v}_k$ which is in the span of the rest.

(warning: this does not mean that every vector is a linear combination of the others.)

proof: suppose $\{\vec{v}_1, \dots, \vec{v}_m\}$ is LD, so there is some non-trivial solution to the dependence equation

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_m\vec{v}_m = \vec{0}$$

for not all $a_i = 0$. Let's say $a_1 \neq 0$, then

$$\vec{v}_1 = \frac{-a_2}{a_1}\vec{v}_2 + \dots + \frac{-a_m}{a_1}\vec{v}_m,$$

$$\Rightarrow \vec{v}_1 \in \text{Span}\{\vec{v}_2, \dots, \vec{v}_m\}.$$

suppose $\vec{v}_n \in \text{span}\{\vec{v}_1, \dots, \vec{v}_{n-1}\}$, or $\vec{v}_n = b_1\vec{v}_1 + \dots + b_{n-1}\vec{v}_{n-1}$. Then,

$$b_1\vec{v}_1 + b_2\vec{v}_2 + \dots + b_{n-1}\vec{v}_{n-1} + (-1)\vec{v}_n = \vec{0}$$

and the coefficient of $\vec{v}_n$ is $-1 \neq 0 \Rightarrow \{\vec{v}_1, \dots, \vec{v}_m\}$ is LD.

$\square$

* Remark: Any linearly dependent spanning set can be reduced.

* Theorem (8.2): Reducing spanning sets

suppose $W = \text{span}\{\vec{v}_1, \dots, \vec{v}_m\}$ is a subspace of V. If $\vec{v}_1 \in \text{span}\{\vec{v}_2, \dots, \vec{v}_m\}$ then

$$W = \text{span}\{\vec{v}_2, \dots, \vec{v}_m\}.$$

In other words, we can decrease the size of any linearly dependent spanning set, until we reach a linearly independent subset.

* Theorem (8.3): Enlarging linearly independent sets.

Suppose $\{\vec{v}_1, \dots, \vec{v}_m\}$ is a LI subset of a subspace W. For any $\vec{v} \in W$, we have

$$\{\vec{v}, \vec{v}_1, \dots, \vec{v}_m\} \text{ is LI} \iff \vec{v} \notin \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_m\}$$

Proof: Suppose $\{\vec{v}, \vec{v}_1, \dots, \vec{v}_m\}$ is LI, then from the previous theorem, no element of it can be a linear combination of the rest, or $\vec{v}$ can't be a linear combination of $\vec{v}_1, \dots, \vec{v}_m$. Therefore,

$$\vec{v} \notin \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_m\}.$$

Conversely, suppose $\vec{v} \notin \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_m\}$, then we want to decide if the dependence equation

$$a_0\vec{v} + a_1\vec{v}_1 + \cdots + a_m\vec{v}_m = \vec{0},$$

could have any non-trivial solution. If $a_0 \neq 0$, then we could solve it for $\vec{v}$ as a linear combinations of others, but our hypothesis is that this is not the case; therefore, $a_0 = 0$. Hence,

$$a_1\vec{v}_1 + \cdots + a_m\vec{v}_m = \vec{0},$$

which is the dependence equation for $\{\vec{v}_1, \dots, \vec{v}_m\}$, which is LI. Therefore, $\{\vec{v}, \vec{v}_1, \dots, \vec{v}_m\}$ is LI. $\square$

In other words, we can increase the size of any linearly independent set, so long as it does not span our vector space.

Examples:

(1) $\{x^2, 1+2x\} \subseteq \mathbb{P}_3$ is LI, then so is the set $\{x^3, 1+2x, x^3\}$, since $x^3 \notin \text{span}\{x^2, 1+2x\}$.

(2) $\left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right\}$ is LI, since $\left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}$ is also LI;

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \notin \text{span}\left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right\},$$

or
$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq a\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$