Chapter 9 : Basis and Dimension

From last time:

1) Any spanning set which is linearly dependent can be reduced (without changing the span), by removing a vector which is in the span of the rest.

2) Any linearly independent set in W, which does not span W can be made into a larger linearly independent set in W, by throwing in a vector which is not in the span of the set.

Example: the set $\{(1,2,1,1), (1,3,5,6)\}$ is LI. Let's find a bigger LI containing them:

$$\text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 5 \\ 6 \end{pmatrix} \right\} = \left\{ a\begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 3 \\ 5 \\ 6 \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} a+b \\ 2a+3b \\ a+5b \\ a+6b \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$$

we have to choose a vector $\vec{v} = (x_1, x_2, x_3, x_4)$, which does not belong to this spanning set. If we try $(1,0,0,0)$ for instance, we see that $\{(1,2,1,1), (1,3,5,6), (1,0,0,0)\}$ is LI

Example: $\{(1,0), (0,1)\}$ is LI & $\text{span}\{(1,0),(0,1)\} = \mathbb{R}^2$. So, There is no $\vec{v} \in \mathbb{R}^2$, such that $\{(1,0), (0,1), \vec{v}\}$ be LI.

Fact: Any set of 3 or more vectores in $\mathbb{R}^2$ is LD.

Note: Any LI set in $\mathbb{R}^2$ has at most two vectors, and any spanning set of $\mathbb{R}^2$ has at least two vectors. Therefore, every linearly independent spanning set has exactly two vectors!

So far:

If S is a spanning set, then any bigger set containing S has to be LD. And if instead S is a linearly independent set, then no proper subset could span the whole space.

* Theorem (9.1): (LI sets are never bigger than spanning sets).

If a vector space V can be spanned by n vectors, then any linearly independent subset has at most n vectors.

or,

if V has a subset of m linearly independent vectors, then any spanning set has at least m vectors.

or,

the size of any linearly independent set in $V \leq$ the size of any spanning set of V,

or,

suppose a vector space V can be spanned by n vectors. If any set of m vectors in V is linearly independent, then $m \leq n$.

Examples :

(1) $\mathbb{R}^3$ is spanned by three vectors, (e.g. $(1,0,0), (0,1,0), (0,0,1)$).
So, any set in $\mathbb{R}^3$ with 4 or more vectors is LD.

(2) $M_{2\times2}(\mathbb{R})$ is spanned by 4 vectors. So, any set of 5 or more 2×2 matrices is LD.

(3) The set of 2x2 diagonal matrices is spanned by two vectors, so any set of 3 or more 2x2 diagonal matrices is LD.

⊛ The theorem talks about the maximum number of linearly independent vectors in any vector space, including subspaces.

★ Basis of a vector Space :

Definition: A set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ of vectors in V is called a "basis" of V if:

(1) $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ is linearly independent, and

(2) $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ spans V.

Equivalently,

- A "basis" is a linearly independent spanning set of V.
- A "basis" is a biggest possible linearly independent set in V.
- A "basis" is a smallest possible spanning set of V.

Examples :

(1) $\{(1,0), (0,1)\}$ is a basis of $\mathbb{R}^2$.

(2) $\{1, x, x^2\}$ is a basis of $\mathbb{P}_2$.

(3) $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is a basis of $M_{2\times 2}(\mathbb{R})$.

* Theorem (9.2): (All bases have the same size).

If $\{\vec{v}_1, \dots, \vec{v}_m\}$ and $\{\vec{w}_1, \dots, \vec{w}_k\}$ are two bases of a vector space V, then $m = k$ (All bases have the same number of vectors).

Proof: Since $\{\vec{v}_1, \dots, \vec{v}_m\}$ spans V and $\{\vec{w}_1, \dots, \vec{w}_k\}$ is LI, then $m \geq k$. Similarly $k \geq m$, then $m = k$. $\square$

Dimension of a vector space:

Definition: If V has a finite basis $\{\vec{v}_1, \dots, \vec{v}_n\}$, then the "dimension" of V is n, the number of vectors in that basis.

$$\boxed{\dim(V) = n}$$

We also say that V is finite-dimensional. If V doesn't have a finite basis, then V is infinite-dimensional.

Examples:

(1) $\dim(\mathbb{R}^2) = 2$,

(2) $\dim(\mathbb{P}_2) = 3$,

(3) $\dim(M_{2\times 2}(\mathbb{R})) = 4$,

(4) The set $\{(1,0,0,\dots,0),(0,1,0,\dots,0),(0,0,1,\dots,0),\dots,(0,0,\dots,1)\}$ of vectors in $\mathbb{R}^n$ is linearly independent and spans $\mathbb{R}^n$ and so $\dim(\mathbb{R}^n)=n$.

(5) The set $\{1,x,x^2,\dots,x^n\}$ is linearly independent and spans $\mathbb{P}_n$ and so $\dim(\mathbb{P}_n)=n+1$.

(6) Consider the set of $m\times n$ matrices $\{E_{ij} \mid 1\leq i\leq m, 1\leq j\leq n\}$, where E_{ij} is the matrix with zeros everywhere except for a 1 at the (i,j)th position. These are linearly independent, since

$$\sum_{i,j} a_{ij}E_{ij}$$

is the matrix with a_{ij} at entry (i,j). The dependence relation has only the trivial solution. They span $M_{m\times n}(\mathbb{R})$ and so this set is a basis of $M_{m\times n}(\mathbb{R})$. Therefore,

$$\dim(M_{m\times n}(\mathbb{R}))=m\times n.$$

(7) The vector space $\mathbb{P}$ of all polynomial functions is infinite-dimensional since for any n the set $\{1,x,x^2,\dots,x^n\}$ is linearly independent and because a basis must be larger than any linearly independent sets, we cannot find a finite basis for $\mathbb{P}$.

(8) The vector space $F(\mathbb{R})$ is also infinite-dimensional.

(9) Consider $L = \{ A \in M_{2\times 2}(\mathbb{R}) \mid Tr(A) = 0 \}$

$$L = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

$$\Rightarrow a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Leftrightarrow a = b = c = 0.$$

So, $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ is linearly independent and also spans L, or it is a basis of $L \Rightarrow \dim(L) = 3$.

(10) Find a basis for $W = \{1, \sin x, \cos x\}$, a subspace of $F(\mathbb{R})$.

Actually, $\{1, \sin x, \cos x\}$ is a spanning set for W and it is also linearly independent, so it is a basis of $W \Rightarrow \dim(W) = 3$.

(11) $U = \{(x, y, z) \mid x + z = 0\}$: find the basis.

$$U = \{(x, y, -x) \mid x, y \in \mathbb{R}\} = \text{Span}\{(1,0,-1), (0,1,0)\}$$

So, $\{(1,0,-1), (0,1,0)\}$ spans U and it is LI, so a basis.

Thus, $\dim(U) = 2$.

Remark: The zero space $\{\vec{0}\}$ has no basis, because no LI set of vectors can contain the zero vector $\vec{0}$. Therefore, an "empty" set of vectors is a basis of $\{\vec{0}\}$, and $\dim\{\vec{0}\} = 0$. Thus, the statement that "the dimension of a vector space is the numbe of vectors in a basis" holds even for the zero space.